Probability Theory on Vector Spaces III

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Probability Theory on Vector Spaces III
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"With each passing biennium, the subject of probability in vector spaces makes more and more impressive gains"
Anatole Beck

PREFACE

The third conference on Probability Theory on Vector Spaces took place in Lublin (Poland) during August of 1983. This conference was sponsored by the Maria Skłodowska-Curie University at Lublin and was organized by the following committee: Z. Rychlik, D. Szynal (chairman) and A. Weron.

This volume contains 26 contributions and complements the material in the two earlier volumes, Springer's Lecture Notes in Math. vol. 656(1978) and vol. 828(1980). We completely agree with Professor Beck's statement taken from the introduction to the proceedings of the conference on Probability in Banach spaces III, Springer's Lecture Notes in Math. vol. 860(1981). His first conference organized in Oberwolfach in 1975 motivated us to organize our own meetings, however we have never intended to compete with Professor Beck's conferences but rather to extend the influence of this new theory. Since there are vector spaces, being natural spaces of sample paths of stochastic processes, which are no longer Banach it is desirable to develop probability theory on general vector spaces.

Two most popular topics of our 1980 conference: stable measures and multidimensional stochastic processes are also present in this volume. The 60th anniversary of Paul Lévy's paper "Théorie des erreurs, La Loi de Gauss et les Lois exceptionnelles", Bull. Soc. Math. France 52(1924), 49-85, initiating the theory of stable distributions is celebrated by four contributions (Hazod, Linde, Rajput & Rama-Murthy, and Weron). Seven papers (Dettweiler, Ferreyra, Leskow, Niemi, Pourahmadi & Salehi, Russek and Shonkwiler) are devoted to vector valued processes and Hilbert space methods in stochastic processes. Readers interested in this subject should consult also the Pesi Masani volume "Harmonic Analysis and Prediction", North-Holland 1983, edited by V. Mandrekar and H. Saléhi. Different problems related to limit theorems on Hilbert, Orlicz, Banach or even Polish spaces are studied by (Heinkel, Inglot & Jurlewicz, Ledoux, Rychlik, Rychlik & Szyszkowski, and Szyznal & Kuczmaszewska). A new feature of this conference are papers (Henze, Goldstein & Łuczak, and Jajte) on ergodic theorems for von Neumann algebras.

Aleksander Weron

Baton Rouge, April 1984
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REMARKS ON RANDOM FUNCTIONAL SPACES

By
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The purpose of the present paper is to give an operator characterization of random measures. This characterization follows from a representation theorem for random elements with values in a dual space of a separable LB-space.

We assume that \((E, \{i_n: E \to E \}_{n=1}^{\infty})\) is a strict inductive limit of an inductive system of separable Banach spaces \(\{E_n\}_{n=1}^{\infty}\), \(\{i_n^m: E_m \to E_n \}_{m \leq n}\). By \(E_s^*, E_b^*\) we denote the duals of \(E\) with the topology of simple or bounded convergence, respectively. We will need two results concerning the canonical bilinear form \(E \times E_s^* \times x^* \to \times (x)(x) \in R\):

(i) \(E \times E_s^* \to R\) is \((\mathcal{M}_s, \mathcal{N}_s)\)-hypocontinuous, where \(\mathcal{M}_s\) is the class of all finite subsets of \(E\), and \(\mathcal{N}_s\) the class of all bounded (= equicontinuous = relatively compact) subsets of \(E_s^*\) (it means \(B: E \times E_s^* \to R\) is separately continuous; for every \(M \in \mathcal{M}_s\), and every neighbourhood \(W\) of zero in \(R\) there exists a neighbourhood \(V\) of zero in \(E_s^*\) such that \(B(M, V) \subseteq \omega\) and the same if we replace \(\mathcal{M}_s\) by \(\mathcal{N}_s\));

(ii) \(E \times E_b^* \to R\) is \((\mathcal{M}_b, \mathcal{N}_s)\)-hypocontinuous, where \(\mathcal{M}_b\) is the
class of all bounded subsets of $E$.

Let $S$ be a locally compact Polish space, and $P$ be a Radon probability measure on $S$. If $X$ is a linear topological space, then by $\text{Mes}(S,X)$ we denote the linear topological space of all $P$-measurable mappings $S \rightarrow X$ considered with the topology of convergence in probability. Let $\text{Mes}(S,X)$ denote the quotient linear space of $\text{Mes}(S,X)$ by the subspace of all $P$-negligible mappings.

First we will give an operator characterization of elements which belong to $\text{Mes}(S,\mathbb{E}^*)$. We begin with two lemmas.

Lemma 1. If $\xi \in \text{Mes}(S,\mathbb{E}^*)$, then there exists a locally countable family $\mathcal{R}$ of distinct compact subsets \(\bigcup_{K \in \mathcal{R}} K\) is $P$-negligible, and a continuous linear operator $\varphi : E \rightarrow \prod_{K \in \mathcal{R}} C(K)$ such that the diagram commutes.

Here $Q(\xi)$ is defined by $Q(\xi)(x) = \xi(x)$ for $x \in E$ ($\xi(x)$ means its equivalence class in fact), $C(K)$ is a Banach space of all continuous functions on $K$, and $w$ is a continuous linear operator such that the diagram commutes for every $K \in \mathcal{R}$.

Proof: $\xi(x)$ belongs to $\text{Mes}(S,\mathbb{E})$ because the form $E^* \ni x^* \mapsto x^*(x)$ is continuous for every $x \in E$. It follows from measurability of $\xi$ that the class of all compact sets $K \subset S$ such that $\xi|_K$ is continuous is $P$-dense, and hence there exists a subclass $\mathcal{R}$ of this above satisfying (1). Consequently the diagram
Continuity of the diagonal arrow follows from \((\mathcal{M}_s, \mathcal{N}_s)\)-hypocontinuity of the form \(B: E \times E^* \to \mathbb{R}\). If \(V \subset C(K)\) is a neighborhood of zero, then there exists \(r > 0\) such that the ball \(D_r\) is contained in \(V\). Because \(\xi(K) \subset E^*_s\) is compact, it follows from the above mentioned hypocontinuity that there exists a neighborhood of zero \(U \subset E\) such that \(B(U, \xi(K)) \subset (-r, r)\). It means that \(\alpha(\xi|_K)(U) \subset D_r \subset V\). Now the diagram commutes

and we have the desired result.

**Lemma 2.** If \(\Phi: E \to \text{Mes}(S, \mathbb{R})\) is such that

there exist a family \(\mathcal{R}\) satisfying (1) and a continuous linear operator \(\psi: E \to \prod_{K \in \mathcal{R}} C(K)\) such that \(\Phi = w \cdot \psi\),

then \(\Phi = \alpha(\xi)\) for some \(\xi \in \text{Mes}(S, E^*)\).

**Proof:** Every \(s \in \bigcup_{K \in \mathcal{R}} K\) determines uniquely \(K_s \in \mathcal{R}\) such that \(s \in K_s\). Let \(\xi(s)\) be a composition

\[
E \xrightarrow{\psi} \prod_{K \in \mathcal{R}} C(K) \xrightarrow{\text{proj}} C(K_s) \xrightarrow{\delta_s} \mathbb{R}, \quad \delta_s(f) = f(s).
\]

If we put \(\xi(s) = 0\) for \(s \in S \setminus \bigcup_{K \in \mathcal{R}} K\), then \(\xi\) is defined for all \(s \in S\). Since for every \(K \in \mathcal{R}\)
it follows that \( \xi|_K: K \to E_s^* \) is continuous and hence \( \xi: S \to E_s^* \) is measurable.

The equality \( \Phi = \alpha(\xi) \) is a consequence of the commutativity of the diagram

\[
\begin{array}{ccc}
\text{Mes}(S, \mathbb{R}) & \xrightarrow{\Phi} & \\
\bigoplus_{K \in \mathcal{K}} \text{Mes}(K, \mathbb{R}) & \xrightarrow{\phi} & \text{Mes}(S, \mathbb{R})
\end{array}
\]

\[
\begin{array}{ccc}
\text{C}(K) & \xrightarrow{\text{proj}} & \\
\bigoplus_{K \in \mathcal{K}} \text{C}(K) & \xrightarrow{\phi} & \text{Mes}(K, \mathbb{R})
\end{array}
\]

Let \( \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) denote the linear space of operators satisfying (2).

**Corollary 1.** The linear spaces \( \text{Mes}(S, E_s^*) \) and \( \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) are isomorphic.

**Proof:** Let \( \pi \) denote the projection \( \text{Mes}(S, E_s^*) \to \text{Mes}(S, E_s^*) \). By above lemmas there exist two mappings: \( \alpha: \text{Mes}(S, E_s^*) \to \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) and \( \beta: \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \to \text{Mes}(S, E_s^*) \) with properties

\[
(\alpha \cdot \beta)(\Phi) = \Phi \text{ for every } \Phi,
\]

\[
(\beta \cdot \alpha)(\xi) = \xi \text{ a.e. for every } \xi,
\]

\( \xi = \eta \text{ a.e. implies } \alpha(\xi) = \alpha(\eta). \)

It follows that the diagram

\[
\begin{array}{ccc}
\text{Mes}(S, E_s^*) & \xrightarrow{\alpha} & \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \text{id} \to \mathcal{L}(E, \text{Mes}(S, \mathbb{R}))
\end{array}
\]

\[
\begin{array}{ccc}
\text{Mes}(S, E_s^*) & \xrightarrow{\pi} & \alpha \text{id} \to \text{Mes}(S, E_s^*)
\end{array}
\]

commutes, and this completes the proof.

We study now operator representations of random elements in \( E_b^* \). By compact operator we mean here an operator which all bounded sets maps into relatively compact sets.

**Lemma 3.** If \( \xi \in \text{Mes}(S, E_b^*) \), then there exist a family \( \mathcal{K} \) satisfying (1) and a compact linear operator \( \psi: E \to \bigoplus_{K \in \mathcal{K}} \text{C}(K) \) such that
\( \Omega(\xi) = \omega \circ \varphi \).

Proof: Since the factorization \( \Omega(\xi) = \omega \circ \varphi \) we obtain in the same way as in lemma 1, it is sufficient to prove compactness of \( \varphi \).

For every \( K \in \mathcal{K} \) the operator \( \Omega(\xi|_K): E \rightarrow C(K) \) is continuous (by the same arguments as in lemma 1) and hence if \( V \in \mathcal{M}_b \), then \( \Omega(\xi|_K)(V) \) is bounded in \( C(K) \). We shall show that \( \Omega(\xi|_K)(V) \) is equicontinuous. By \((\mathcal{M}_b, \mathcal{M}_b)\)-hypocontinuity of \( B: E \times E^*_b \rightarrow \mathbb{R} \) we obtain that for every \( \varepsilon > 0 \) there exists a neighbourhood of zero \( W \subset E^*_b \) such that \( B(V, W) < (-\varepsilon/2, \varepsilon/2) \). Since the family

\[ \{\xi^{-1}(\xi(s)+W)\}_{s \in K} \]

is an open cover of the compact metric space \((K, d)\), it follows that there exists \( \delta > 0 \) such that the open cover \( \{s \in K: d(s, s') < \delta\}_{s \in K} \) refines \( \{\xi^{-1}(\xi(s)+W)\}_{s \in K} \). Then if \( d(s, s') < \delta \) we have \( s, s' \in \xi^{-1}(\xi(s)+W) \) for some \( s_0 \in K \) and it means that \( \xi(s) - \xi(s_0), \xi(s') - \xi(s_0) \in W \). Thus for every \( x \in V \) we have

\[ |\Omega(\xi)(x(s)) - \Omega(\xi)(x(s'))| \leq |B(x, \xi(s) - \xi(s_0))| + |B(x, \xi(s') - \xi(s_0))| < \varepsilon \]

Since \( \Omega(\xi|_K)(V) \) is bounded and equicontinuous, it must be relatively compact in \( C(K) \). By Tychonoff theorem we obtain that the image of \( V \) in \( \bigcap_{K \in \mathcal{K}} C(K) \) is relatively compact.

Lemma 4. If \( \Phi: E \rightarrow \text{Mes}(S, \mathbb{R}) \) is such that there exist a family \( \mathcal{M} \) satisfying (1) and a compact linear operator \( \varphi: E \rightarrow \bigcap_{K \in \mathcal{K}} C(K) \) such that \( \Phi = \omega \circ \varphi \),

then \( \Phi = \Omega(\xi) \) for some \( \xi \in \mathcal{M} \).

Proof: Since an operator \( \xi: S \rightarrow E^* \) such that \( \Phi = \Omega(\xi) \) we define in the same way as in lemma 2, it is sufficient to prove measurability of \( \xi|_K: K \rightarrow E^*_b \).

Let \( K \in \mathcal{K} \). We shall show that \( \xi|_K: K \rightarrow E^*_b \) is continuous. If \( U \subset E^*_b \) is a neighbourhood of zero, then there exist \( V \in \mathcal{M}_b \) and \( \varepsilon > 0 \) such that \( \{x \in E^*: x^*(V) < (-\varepsilon, \varepsilon)\} \subset U \). Since the operator \( \Omega|_K: E \rightarrow \bigcap_{K \in \mathcal{K}} C(K) \rightarrow C(K) \) is compact, \( \Omega|_K(V) \) is equicontinuous and hence for some \( r > 0 \) and for all \( s, s' \in K \) such that \( d(s, s') < r \) we have \( |\xi(s)x) - \xi(s'x)| < \varepsilon \) for \( x \in V \) or equivalently \( \xi(s) - \xi(s') \) belongs to \( \{x^*: x^*(V) < (-\varepsilon, \varepsilon)\} \subset U \).

Let \( \overline{\mathcal{C}}(E, \text{Mes}(S, \mathbb{R})) \) denote a linear space of operators satisfying (3). Then by the same method as Corollary 1 we can obtain

Corollary 2. The linear spaces \( \text{Mes}(S, E^*_b) \) and \( \overline{\mathcal{C}}(E, \text{Mes}(S, \mathbb{R})) \)
are isomorphic.
We shall show that isomorphisms from Corollaries 1 and 2 are homeomorphisms with respect to suitable topologies.

In \( \mathcal{L}(E, \text{Mes}(S, E)) \) and \( \mathcal{L}(E, \text{Mes}(S, E^*)) \) we consider topology of convergence in probability: its neighbourhood base of zero is formed by the sets \( \{ \xi : P(\xi(x) < \varepsilon) \} \), where \( \varepsilon > 0 \) and \( U \) is a neighbourhood of zero in \( E^* \) or \( E_b^* \), respectively.

In \( \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) and \( \mathcal{F}(E, \text{Mes}(S, \mathbb{N})) \) we consider some operator topologies: their neighbourhood bases of zero are formed by the sets \( \{ \Phi : P(\Phi(x) > \delta) > 1 - \varepsilon \} \), where \( W \) is a neighbourhood of zero in \( \text{Mes}(S, \mathbb{R}) \) and \( V \in \mathcal{M}_S \) in the case of \( \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) or \( V \in \mathcal{M}_b \) in the case of \( \mathcal{F}(E, \text{Mes}(S, \mathbb{R})) \).

Let \( \{ \Phi : x \in V \} \) be a neighbourhood of zero in \( \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) or in \( \mathcal{L}(E, \text{Mes}(S, \mathbb{N})) \). Then \( \{ \xi : P(\xi(x) < \delta) > 1 - \varepsilon \} \) is a neighbourhood of zero in \( \text{Mes}(S, E^*_S) \) or in \( \text{Mes}(S, E_b^*) \) and we have

\[
\forall x \in V \quad P(\Phi(x) > \delta) \leq P(\Phi(x) > \delta) = P(\Phi(x) < \delta)
\]

and hence the mappings \( \text{Mes}(S, E^*_S) \rightarrow \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) and \( \text{Mes}(S, E_b^*) \rightarrow \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \) are continuous.

Since \( P(\exists x \in V \mid \xi(x) > \delta) \leq \sum_{x \in V} P(\xi(x) > \delta) = \sum_{x \in V} P(\tilde{\mu}(\xi(x)) > \delta) \), we have for \( V \in \mathcal{M}_S \)

\[
\mathcal{A}^{-1}(\Phi) \subset \mathcal{A}^{-1}(\tilde{\mu}(\xi(x)) > \delta) \leq \varepsilon / \text{card } V
\]

and hence the mapping \( \mathcal{L}(E, \text{Mes}(S, \mathbb{R})) \rightarrow \text{Mes}(S, E^*_S) \) is continuous.

For the proof of continuity of the mapping \( \mathcal{F}(E, \text{Mes}(S, \mathbb{R})) \rightarrow \text{Mes}(S, E_b^*) \) we need following lemmas.

**Lemma 5.** If \( E \) is a normed space, then \( \mathcal{A}^{-1} \) is continuous.

**Proof:** Let \( D \) denote the ball \( \{ x \in E : \|x\| < 1 \} \). By our assumption \( E_b^* \) is a Banach space with a norm sup. Thus the topologies in \( \text{Mes}(S, E^*_b) \) and \( \mathcal{F}(E, \text{Mes}(S, \mathbb{R})) \) are consistent with metrics

\[
d_1(\xi, \eta) = \int_{x \in D} \frac{\sup_{x \in D} |\tilde{\mu}(\xi(x) - \eta(x))|}{1 + \sup_{x \in D} |\tilde{\mu}(\xi(x) - \eta(x))|} \, dP \quad \text{and} \quad d_2(\Phi, \Psi) = \int_{E_b^*} \frac{|\Phi(x) - \Psi(x)|}{1 + |\Phi(x) - \Psi(x)|} \, dP,
\]
respectively. Since for every \( \varepsilon > 0 \) there exists \( x_0 \in D \) such that
\[
\sup \{|\xi(x) - \eta(x)| : x \in D\} \leq \varepsilon + |\xi(x_0) - \eta(x_0)|,
\]
we have
\[
\frac{\sup \{|\xi(x) - \eta(x)|\}}{1 + \sup \{|\xi(x) - \eta(x)|\}} \leq \varepsilon + \frac{|\xi(x_0) - \eta(x_0)|}{1 + |\xi(x_0) - \eta(x_0)|}
\]
and hence \( d_1(\xi, \eta) \leq \varepsilon + d_2(\xi, \eta) \) for every \( \varepsilon > 0 \). Thus replacing \( \xi \) by \( \tilde{\Phi}^{-1}(\Phi) \) and \( \eta \) by \( \tilde{\Psi}^{-1}(\Psi) \) we obtain
\[
d_1(\tilde{\Phi}^{-1}(\Phi), \tilde{\Psi}^{-1}(\Psi)) \leq d_2(\Phi, \Psi).
\]

Lemma 6. The diagram

\[
\begin{array}{ccc}
\text{Mes}(S, E_b^\kappa) & \xrightarrow{\tilde{\Phi}^{-1}} & \tilde{\mathcal{E}}(E, \text{Mes}(S, R)) \\
\downarrow \delta & & \downarrow \mathcal{T} \\
\text{Mes}(S, (E_n)_b^\kappa) & \xrightarrow{\tilde{\Psi}^{-1}} & \tilde{\mathcal{E}}(E_n, \text{Mes}(S, R))
\end{array}
\]

commutes and the mappings \( \mathcal{T}, \tilde{\Phi}^{-1}, \delta \) are continuous.

Here \( \tilde{\Phi}(\xi) = i_n^\kappa \Phi \xi \) for \( \xi \in \text{Mes}(S, E_b^\kappa) \)
\( \mathcal{T}(\Phi) = \Phi^1_n \) for \( \Phi \in \tilde{\mathcal{E}}(E, \text{Mes}(S, R)) \).

Proof: Lemma 5 implies continuity of \( \tilde{\Phi}^{-1} \).

Continuity of \( i_n^\kappa : E_b^\kappa \rightarrow (E_n)_b^\kappa \) implies that \( \tilde{\Phi}(\xi) \in \text{Mes}(S, (E_n)_b^\kappa) \).

If \( \{\eta \in \text{Mes}(S, (E_n)_b^\kappa) : P(\eta \in U) \geq 1 - \varepsilon\} \) is a neighbourhood of zero, then \( i_n^\kappa^{-1}(U) \) is a neighbourhood of zero in \( E_b^\kappa \),
\[
\{\xi \in i_n^\kappa^{-1}(U)\} = \{i_n^\kappa \xi \in U\} = \{\tilde{\Phi}(\xi) \in U\}
\]
and \( \delta \{\xi : P(\xi \in i_n^\kappa^{-1}(U)) \geq 1 - \varepsilon\} \subseteq \{\eta : P(\eta \in U) \geq 1 - \varepsilon\} \).

Thus \( \delta \) is continuous.

Remark: In fact we can obtain that
\[
\{\xi : P(\xi \in i_n^\kappa^{-1}(U)) \geq 1 - \varepsilon\} = \delta^{-1}\{\eta : P(\eta \in U) \geq 1 - \varepsilon\} \quad (4)
\]

Continuity of \( \mathcal{T} \) is obvious because every bounded subset of \( E_n \) is bounded in \( E \).

Let \( \{\xi : P(\xi \in U) \geq 1 - \varepsilon\} \) be a neighbourhood of zero in \( \text{Mes}(S, E_b^\kappa) \)
where \( U \) is of the form \( \{x^\kappa \in E^\kappa : x^\kappa(V) \subseteq (-\delta, 0)\} \) for \( \delta > 0 \), and \( V \in \mathcal{T}_b^\delta \). Then there exists \( n \in \mathbb{N} \) such that \( V \subseteq E_n \), and we have
\[
i_n^\kappa\left\{x^\kappa \in E_n^\kappa : x^\kappa(V) \subseteq (-\delta, 0)\right\} \neq U . \quad (4)
\]
By (4) we have \( \delta \{\xi : P(\xi \in U) \geq 1 - \varepsilon\} \neq \delta^{-1}\{\eta : P(\eta \in V) \subseteq (-\delta, 0)\} \geq 1 - \varepsilon\} \) and hence
$\mathcal{A}^{-1}\{\xi \in \mathcal{P} : P(\xi \in U) \geq 1 - \varepsilon\} = \left(\bigcap_{n} \mathcal{A}^{-1}\{\eta : P(\eta(V) \in (-\delta, \delta)) \geq 1 - \varepsilon\}\right)

is an open subset of $\mathcal{L}(E, \text{Mes}(S, \mathbb{R}))$.

Now we are able to formulate the main result.

**Theorem.** The mappings $\text{Mes}(S, E^*_s) \ni \xi \mapsto (x \mapsto \xi(x)) \in \mathcal{L}(E, \text{Mes}(S, \mathbb{R}))$, and $\text{Mes}(S, E^*_b) \ni \xi \mapsto (x \mapsto \xi(x)) \in \mathcal{L}(E, \text{Mes}(S, \mathbb{R}))$ are topological isomorphisms.

We can assume that $E$ is a space $\mathcal{M}(T)$ of all continuous functions with compact supports defined on some locally compact Polish space $T$. Then $E^*_s = \mathcal{M}_s(T)$ and $E^*_b = \mathcal{M}_b(T)$ is the Radon measures space with vague or strong topology, respectively.

**Corollary.** $\text{Mes}(S, \mathcal{M}_s(T)) \cong \mathcal{L}(\mathcal{M}(T), \text{Mes}(S, \mathbb{R}))$

$\text{Mes}(S, \mathcal{M}_b(T)) \cong \mathcal{L}(\mathcal{M}(T), \text{Mes}(S, \mathbb{R}))$

References

STOCHASTIC INTEGRAL EQUATIONS AND DIFFUSIONS ON BANACH SPACES

by E. DETTWEILER

Introduction

Let \((Z,\mathcal{F})\) be a measurable space and let \(\xi\) be an independently scattered Gaussian random measure on \(\mathbb{R}_+ \times Z\) whose control measure \(\beta := \mathbb{E} \xi^2\) is given by

\[
\beta((s,t] \times A) := \int_s^t \varphi_r(A) \, dr \quad (0 \leq s < t, A \in \mathcal{F}),
\]

where \(\varphi\) is a \(\mathcal{C}\)-finite kernel from \(\mathbb{R}_+\) to \(Z\). We study stochastic integral equations on a separable Banach space \(E\) which are of the following type:

\[
(I) \quad X_t = x + \int_0^t g(r, X_r) \, dr + \int_0^t \int_Z G(r, z, X_r) \, \xi(dr, dz),
\]

where \(g, G\) are suitable \(E\)-valued functions defined on \(\mathbb{R}_+ \times E\) and \(\mathbb{R}_+ \times Z \times E\) respectively.

In [4] it was proved that under fairly general assumptions an \(E\)-valued Gaussian process with independent increments (not necessarily time homogeneous) can be represented (up to stochastic equivalence) as a stochastic integral relative to a Gaussian random measure \(\xi\) on \(\mathbb{R}_+ \times E\). The infinite dimensional Wiener process gives another example of a Gaussian random measure in taking \(Z := \mathbb{R}_+\).

These two remarks may serve as a sort of motivation for considering stochastic integral equations of the form (I).

To give a precise meaning to the second integral in (I) we introduce in the first paragraph spaces of functions \(f: \mathbb{R}_+ \times Z \rightarrow E\), for which the stochastic integrals \(\int_s^t \int_Z f \, d\xi\) exist as \(E\)-valued random vectors and for which \((\int_s^t \int_Z f \, d\xi)_{t \geq s}\) is an \(E\)-valued martingale relative to the natural filtration given by \(\xi\). It turns out that this problem is connected with the geometry of \(E\). In the special case that \(E\) has an equivalent 2-uniformly smooth norm (and only in this case) one knows that every predictable function in \(L^2(\mathcal{P} \circ \mathcal{F}, E)\) is \(\xi\)-integrable (cf. [4]).

If the function \(G\) in (I) has the property that \((t, z, \omega) \mapsto G(t, z, X_t(\omega))\) is \(\xi\)-integrable (in the sense of the first paragraph) for sufficiently many processes \(X\), then one can prove under additional boundedness and Lipschitz conditions that (I) has a solution.

The solutions of the equations (I) define a strong Markov
process on $\Omega := \mathbb{R}_+^* \times E \times \Omega$ with values in $E$ whose local properties are studied in §3. Let $(\mathbb{P}_{t,x})_{t \geq 0, x \in E}$ denote the family of probability measures on $\Omega$ describing the behaviour of $X$. Then $X$ has the following properties:

1. \[ \lim_{h \to 0} h^{-2} \mathbb{E}_{t,x} \left[ \sup_{\|r\| \leq h} \|X_{t+r} - x\|^4 \right] < \infty, \]

2. \[ \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} [X_{t+h} - x] = g(t,x) \]

3. \[ \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} \left[ (X_{t+h} - x)^2 \right] = \int_Z G(t,z,x) \varphi_t^2(dz) \text{ (in } L^2(E)) \]

for all $t \geq 0$ and $x \in E$, and the convergence is uniform on bounded subsets of $\mathbb{R}_+^* \times E$.

These properties of $X$ have several nice consequences. An almost immediate consequence is the following result. For every two times continuously differentiable function $f : E \to F$ (where $F$ is a second Banach space) with uniformly bounded and uniformly continuous second derivative one gets for all $t \geq 0$, $x \in E$

\[ \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} [f(X_{t+h}) - f(x)] = f'(x)(g(t,x)) + \frac{1}{2} f''(x)(Q(t,x)) \]

uniformly on bounded subsets of $\mathbb{R}_+^* \times E$, where $Q(t,x)$ denotes the covariance operator $\int_Z G(t,z,x) \varphi_t^2(dz)$. Another consequence of properties (1.) to (3.) is the following local weak limit result. Let $\mu_h$ denote the distribution of $h^{-1/2}(X_{t+h} - x)$ relative to $\mathbb{P}_{t,x}$. Then $\lim_{h \to 0} \mu_h = \varphi(t,x)$ weakly, where $\varphi(t,x)$ is the Gaussian measure on $E$ with covariance operator $Q(t,x)$.

In the last paragraph we show that the solution $X$ of (I) can also be obtained as the limit of a sequence of Markov chains.

§1 - Stochastic integration relative to Gaussian random fields

Let $(Z, \mathcal{Z})$ denote a measurable space and let $\beta$ be a measure on the product space $(\mathbb{R}_+^* \times Z \times \mathcal{Z})$ with the properties:

(i) If $\beta_t$ denotes the measure on $(Z, \mathcal{Z})$ defined by $\beta_t(C) := \beta([0,t] \times C)$ for all $C \in \mathcal{Z}$, then for all $C \in \mathcal{Z}_0 := \{ C \in \mathcal{Z} : \beta_t(C) < \infty \text{ for all } t \geq 0 \}$

the map $t \mapsto \beta_t(C)$ is continuous.

(ii) There exists a sequence $(C_k)_{k \geq 1}$ in $\mathcal{Z}_0$ with $Z = \bigcup C_k$.

1.1 Definition. Let $\xi := (\xi_t(A))_{t \geq 0, A \in \mathcal{Z}_0}$ be a real valued stochastic process - defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ - with the following properties:
(1) For all \( t \in \mathbb{R}_0^+ \) and \( \omega \in \mathcal{F}_0^+ \), the distribution of \( \xi_t(\omega) \) is \( \mathcal{N}(0, \beta_t(\omega)) \).

(2) For all \( \omega \in \mathcal{F}_0^+ \), the process \( (\xi_t(\omega))_{t \in \mathbb{R}_0^+} \) is a continuous process with independent increments.

(3) For every disjoint sequence \( (A_k)_{k \geq 1} \) in \( \mathcal{F}_0^+ \) and all \( t \in \mathbb{R}_0^+ \) the sequence \( (\xi_t(A_k))_{k \geq 1} \) is independent and \( \xi_t(\bigcup A_k) = \sum \xi_t(A_k) \) \( \mathbb{P} \)-a.s. in case \( \bigcup A_k \in \mathcal{F}_0^+ \).

Then \( \xi \) is called a Gaussian random field with variance measure \( \beta \).

It can be shown (see [3], p. 36, Prop. 2.1 for the idea of the proof) that to any measure \( \beta \) on \( \mathbb{R}_+ \times Z \) with properties (i) and (ii) there exists a Gaussian random field \( \xi \) with \( \beta \) as its variance measure. Gaussian random fields occur naturally in the connection with Banach space valued Gaussian processes with independent increments. Under fairly general conditions (see [4], p. 65, Theorem 2.9) one can prove that a Gaussian process \( (X_t)_{t \in \mathbb{R}_0^+} \) on a Banach space \( E \) of type 2 is stochastically equivalent to a Gaussian process of the form \( (\int x_t(dx))_{t \in \mathbb{R}_0^+} \), where \( x \) is a Gaussian random field on \( \mathbb{R}_+ \times E \).

The most simple example of a Gaussian random field is of course the infinite dimensional Wiener process \( (\xi_t^1, \xi_t^2, \ldots)_{t \in \mathbb{R}_0^+} \). Take \( Z := \mathbb{R}^N \) and \( \beta_t(k) := t \) for all \( t \in \mathbb{R}_0^+ \), \( k \in \mathbb{N} \).

Now let \( (\xi_t)_{t \in \mathbb{R}_0^+} \) be a fixed filtration on \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that for all \( \omega \in \mathcal{F}_0^+ \), the process \( (\xi_t(\omega))_{t \in \mathbb{R}_0^+} \) is adapted to \( (\xi_t)_{t \in \mathbb{R}_0^+} \) and such that for all \( t \in \mathbb{R}_0^+ \) and \( u < v \), the increments \( \xi_v(\omega) - \xi_u(\omega) \) are independent from \( \xi_t \).

Let \( \mathcal{P} \) denote the \( \sigma \)-algebra of predictable sets on \( \mathbb{R}_+ \times Z \) which is by definition generated by the family \( \mathcal{R} \) of all rectangles of the form \( [s,t] \times A \times F \) with \( s < t \), \( A \in \mathcal{F}_0^+ \) and \( F \in \mathcal{F}_Z^+ \).

For a fixed separable Banach space \( E \) we denote by \( \mathcal{E} \) the vector space generated by the \( E \)-valued functions \( x \cdot 1_R \) with \( x \in E \) and \( R \in \mathcal{R} \).

The elements of \( \mathcal{E} \) will be called elementary predictable (\( E \)-valued) functions. For any \( f \in \mathcal{E} \) we can define the stochastic integral \( \int \xi d\xi \) of \( f \) relative to \( \xi \) as follows. \( f \) is a finite sum of the form

\[
\sum_{i,k} x_{ik} 1_{t_k,t_{k+1}} \cdot X_{ik} \cdot F_{ik}
\]

where all rectangles are in \( \mathcal{R} \) and \( t_0 < t_1 < \ldots < t_n \). We now put

\[
\int f d\xi := \int_{\mathbb{R}_+^+ \times Z} f(t,z) \xi(dt,dz) := \sum_{i,k} x_{ik} 1_{P_{ik}} (\xi_{t_{k+1}} (C_{ik}) - \xi_{t_k} (C_{ik})).
\]

By the assumptions on \( \xi \) the stochastic integral is well-defined and \( S: \mathcal{E} \rightarrow L_0(\Omega, \mathcal{F}, \mathbb{P}; E) \) given by \( S(f) := \int f d\xi \) for all \( f \in \mathcal{E} \) is linear.

For the extension of the elementary stochastic integral to larger classes of \( E \)-valued functions one therefore has to look for topologies on \( \mathcal{E} \) such that \( S \) becomes a continuous operator. For this
we define the following function spaces:

\[ \mathcal{W}_2 := L^2(\mathbb{R}^+ \times \mathcal{Z} \times \Omega, \mathcal{F}, \mathbb{P}; P; E) \] — the space of all predictable E-valued functions \( f \) such that \( \| f \|_2 := \left( \mathbb{E}(\int \| f \|^2 \, d\beta) \right)^{1/2} < \infty \),

\[ \mathcal{W}_2^{1,\infty} := \text{the space of all predictable } E\text{-valued functions } f \text{ such that} \]

\[ \int_{[0, \tau] \times \mathcal{Z}} \| f \|^2 \, d\beta < \infty \text{ P-a.s. for all } \tau > 0. \]

\[ \mathcal{W}_2^{1,\infty} \] is given a topology of convergence in probability in the sense that a sequence \( (f_n) \) converges to \( f \) in \( \mathcal{W}_2^{1,\infty} \) if for all \( \tau > 0 \)

\[ \lim_{n \to \infty} \int_{[0, \tau] \times \mathcal{Z}} \| f_n - f \|^2 \, d\beta = 0 \text{ in probability.} \]

It is not difficult to prove that \( \mathcal{W}_2^{1,\infty} \) is exactly the space of all predictable \( f \) for which there exists an increasing sequence \( (\tau_n)_{n \geq 1} \) of stopping times with \( \lim \mathbb{P}[\tau_n \leq t] = 0 \) for all \( t > 0 \) such that the sequence \( (f^1_{[0, \tau_n]} \mathbb{P})_{n \geq 1} \) belongs to \( \mathcal{W}_2^{1,\infty} \). If there is a measure \( \mathcal{P} \) on \((Z, \mathcal{Z})\) such that \( \beta = \lambda \otimes \mathcal{P} \) (\( \lambda := \) Lebesgue measure on \( \mathbb{R}_+ \)), then \( \mathcal{W}_2^{1,\infty} \) is the space of all progressively measurable \( E \)-valued functions with

\[ \int_{[0, \tau] \times \mathcal{Z}} \| f \|^2 \, d\beta < \infty \text{ P-a.s. for all } \tau > 0. \]

Now let us call \( E \) 2-smoothable if there is an equivalent 2-uniformly smooth norm on \( E \). This means that there is a norm \( \| \cdot \| \) on \( E \) equivalent to the given norm such that for a certain constant \( K > 0 \)

\[ \| x + y \|^2 + \| x - y \|^2 \leq 2 \| x \|^2 + K \| y \|^2 \text{ for all } x, y \in E. \]

One knows (see [12]) that a Banach space is 2-smoothable if and only if there exists a constant \( C > 0 \) such that for every \( E \)-valued, square integrable martingale \( (M_n)_{n \geq 1} \) the inequality

\[ (M) \sup_{n \geq 1} \mathbb{E} \| M_n \|^2 \leq C \sum_{k \geq 0} \mathbb{E} \| M_{k+1} - M_k \|^2 \quad (M_0 := 0) \]

holds, for a constant \( C > 0 \) such that \((M)\) holds we will use the notation \( C_s \) and call \( C_s \) a smoothness constant.

For \( f = \sum_{i,k} x_{ik} \mathbb{I}_{k} \mathbb{I}_{t_k \leq t \leq t_{k+1}} \mathbb{C}_{ik} \mathbb{F}_{ik} \in \mathcal{E} \) it is obvious that

\[ \left( \int_{[0, t_k]} f \, d\xi \right)_{k \geq 1} \]

is a martingale relative to \( (\mathcal{F}_k)_{k \geq 1} \). Hence

\[ \mathbb{E} \left( \int f \, d\xi \right)^2 \leq C_s \mathbb{E} \left( \int \| f \|^2 \, d\beta \right) \}

if \( E \) is 2-smoothable, i.e., the operator \( S : \mathcal{E} \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}; E) \) is continuous, if \( \mathcal{E} \) carries the norm induced by \( \mathcal{W}_2^{1,\infty} \). More generally, the following result holds (see [4] for a complete proof):

1.2 Theorem. Suppose \( \beta \not= \infty \). Then \( E \) is 2-smoothable if and only if

\[ S : (\mathcal{E}, \mathcal{W}_2) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}; E) \]

is continuous. Hence \( S \) can be extended
to a continuous linear operator $S: \mathbb{L}_2 \rightarrow L_2(\Omega, \mathcal{F}, P; E)$ if and only if $E$ is 2-smoothable. In that case one can define for all $f \in \mathbb{L}_2$ the stochastic integral of $f$ relative to $\xi$ by

$$\int f \, d\xi := \int_{\mathbb{R}_+} \int_{\mathbb{Z}} f(t, z) \xi(dt, dz) := S(f).$$

From the inequality $E \int \|f\|^2 d\beta \leq C_\beta \mathbb{E}(\|f\|^2 d\beta)$ ($f \in \mathcal{E}$) it follows exactly as in the classical case (cf. e.g. [8], p. 38o) that the following inequality holds for all $f \in \mathcal{E}$:

$$P \left[ \|f\|_2 > \delta \right] \leq C_\delta/\delta^2 + P \left[ \int f^2 d\beta > \varepsilon \right] \text{ for all } \varepsilon, \delta > 0.$$  

This inequality now implies that the stochastic integral can be extended to all of $\mathbb{L}_2^{10c}$ (see [8]):

1.3 Theorem. Suppose that $E$ is 2-smoothable. If $f \in \mathbb{L}_2^{10c}$ and $t > 0$, then for every sequence $(f_n)$ in $\mathcal{E}$ with $\lim \int_{[0, t] \times \mathbb{Z}} f_n^2 d\beta = 0$ in probability the sequence $(\int f_n \, d\xi)$ is a Cauchy sequence in probability. The limit

$$\int_0^t \int_{\mathbb{Z}} f(s, z) \xi(ds, dz) := \lim_{n \to \infty} \int f_n \, d\xi$$

is independent of the special choice of the approximating sequence.

Theorem 1.3 extends in the following sense. For any sequence $(f_n)$ in $\mathbb{L}_2^{10c}$ such that $\lim f_n = f$ (in $\mathbb{L}_2^{10c}$) one has for all $t > 0$

$$\lim_{n \to \infty} \int_0^t \int_{\mathbb{Z}} f_n(s, z) \xi(ds, dz) = \int_0^t \int_{\mathbb{Z}} f(s, z) \xi(ds, dz)$$

in probability. Moreover, it can be proved (again by the classical arguments) that for all $f \in \mathbb{L}_2^{10c}$ the process

$$(\int_0^t \int_{\mathbb{Z}} f(s, z) \xi(ds, dz))_{t > 0}$$

is an a.s. continuous martingale.

If $E$ is not 2-smoothable, then Theorem 1.2 shows that the stochastic integral defined on $\mathcal{E}$ can not be extended to all of $\mathbb{L}_2$. Nevertheless one can extend the elementary stochastic integral also in this general situation always to a larger function space. Let us define the following subspace of $\mathbb{L}_2$

$$\mathbb{L}_2, \xi := \text{the space of all } f \in \mathbb{L}_2 \text{ for which there exists a sequence } (f_n) \text{ in } \mathcal{E} \text{ such that } \lim f_n = f \text{ in } \mathbb{L}_2 \text{ and such that at the same time } (\int f_n \, d\xi) \text{ is a Cauchy sequence in } L_2(\Omega, \mathcal{F}, P; E).$$

It follows from the equality

$$\mathbb{E} \left< \int g d\xi, x' \right>^2 = \mathbb{E}(\int \left< g, x \right>^2 d\beta)$$

valid for all $g \in \mathcal{E}$ and all $x' \in \mathcal{E}'$ (the dual of $E$) that
\[ \int f \, d\xi := \lim \int f_n \, d\xi \] is independent from the special approximating sequence \( (f_n) \). Hence the stochastic integral is well defined for all \( f \in \mathbb{P}_2,\xi \). Moreover, \( \mathbb{P}_2,\xi \) is a Banach space relative to the norm \( \| \cdot \|_{\mathbb{P}_2,\xi} \) defined by
\[ \| f \|_{\mathbb{P}_2,\xi} := (\mathbb{E}(\int \| f \|^2 \, d\beta))^{1/2} + (\mathbb{E} \| f \|^2)^{1/2} \]
for all \( f \in \mathbb{P}_2,\xi \) and hence \( \mathcal{E} \) is strictly contained in \( \mathbb{P}_2,\xi \).

Remark: In the non-2-smoothable case it will be in general a difficult problem to decide whether a given function in \( \mathbb{P}_2 \) belongs to \( \mathbb{P}_2,\xi \). Hence it seems to be interesting to study the space \( \mathbb{P}_2,\xi \) in order to get simpler conditions for \( \xi \)-integrability.

Analogously to the 2-smoothable case the stochastic integral defined for functions in \( \mathbb{P}_2,\xi \) can be extended to a larger space of locally \( \xi \)-integrable functions. Define
\[ \mathbb{P}^{\text{loc}}_{2,\xi} := \text{the space of all } E\text{-valued predictable functions } f \text{ for which there exists an increasing sequence } (T_n) \text{ of stopping times with } \lim P[T_n < t] = 0 \text{ for all } t > 0 \text{ such that } \int_{[0, T_n]} f \, d\xi \text{ is a Cauchy sequence in probability for all } T_n \to 0. \]

Again one can prove that
\[ \int_0^t \int_0^z f(s, z) \xi(ds, dz) := \lim_{n \to \infty} \int f_{[0, T_n]} \, d\xi \] (in probability) is independent of the special sequence \( (T_n) \) and hence the stochastic integral (up to an arbitrary time \( t > 0 \)) is defined for all \( f \in \mathbb{P}^{\text{loc}}_{2,\xi} \).

For later purposes we will now look for conditions under which the stochastic integral of a function is in \( L_p(\Omega, \mathcal{F}, P; E) \) for \( p > 2 \). For this we define
\[ \mathbb{P}_{p,2} := \text{the space of all predictable } E\text{-valued functions } f \text{ such that } \| f \|_{\mathbb{P}_{p,2}} := [\mathbb{E}(\int \| f \|^2 d\beta)^{p/2}]^{1/p} < \infty. \]
\( \mathbb{P}_{p,2} \) is a Banach space relative to the norm \( \| \cdot \|_{\mathbb{P}_{p,2}} \).

1.4 Theorem. Suppose \( \beta \neq 0 \) and \( 2 \leq p \leq 4 \). If (and only if) \( E \) is 2-smoothable, there exists a constant \( C_{s,p} > 0 \) such that for all \( f \in \mathbb{P}_{p,2} \):
\[ \mathbb{E} \| f \|_p^p \leq C_{s,p} \mathbb{E}(\int \| f \|^2 d\beta)^{p/2}. \]

Proof: Suppose first that \( E \) is 2-smoothable. Since \( \mathcal{E} \) is dense in \( \mathbb{P}_{p,2} \), we only have to prove the asserted inequality for functions
\[ f = \sum_{i,k} x_{ik} 1_{[t_k, t_{k+1}]} c_{ik} \xi \quad \in \mathcal{E}. \]
By definition we have
\[ \int f \, d\xi = \sum_{i,k} x_{ik} 1_{p,i k} (\xi_{t_{k+1}}(C_{ik}) - \xi_{t_k}(C_{ik})) , \]
and therefore \( \int f \, d\xi \) defines an \( E \)-valued martingale \((M_j)_{1 \leq j \leq m}\) relative to an obvious filtration \((\mathcal{G}_j)_{1 \leq j \leq m}\) such that
\[ \int f \, d\xi = M_m = \sum_{1 \leq j \leq m} g_j \xi_j \quad \text{and} \quad M_n = \sum_{1 \leq j \leq n} g_j \xi_j \quad (n \leq m) , \]
where the \( g_j \) are \( \mathcal{G}_{j-1} \)-measurable \( E \)-valued functions and the \( \xi_j \) are independent real random variables which are independent from \( \mathcal{G}_{j-1} \) and \( N(0, \lambda_j) \)-distributed with \( \lambda_j > 0 \). Now it follows from [12] (p.346) that there exists a constant \( K > 0 \) depending only on \( E \) such that
\[
\mathbb{E} \left( \sum_{1 \leq j \leq m} g_j \xi_j \right)^p \leq K \left( \sum_{1 \leq j \leq m} \| g_j \|_E^2 \| \xi_j \|_E^2 \right)^{p/2}
\]
\[ = K \mathbb{E} \left[ \sum_{j} \| g_j \|_E^2 \lambda_j + \sum_{j} \| g_j \|_E^2 (\xi_j^2 - \lambda_j) \right]^{p/2}
\]
\[ \leq C \mathbb{E} \left( \sum_{j} \| g_j \|_E^2 \lambda_j \right)^{p/2} + C \mathbb{E} \left( \sum_{j} \| g_j \|_E^2 (\xi_j^2 - \lambda_j) \right)^{p/2}
\]
with \( C := 2^{p/2 - 1} \). We put \( h_j := \| g_j \|_E^2 \) and \( \eta_j := \xi_j^2 - \lambda_j \) (1 \( \leq j \leq m \)). Since \( q := p/2 \leq 2 \) we get
\[
\left( \mathbb{E} \left( \sum_{j} h_j \eta_j \right) \right)^{1/q} \leq \left( \sum_{j} \mathbb{E} \left( h_j \eta_j^2 \right) \right)^{1/2}
\]
and hence
\[
\mathbb{E} \left( \sum_{j} \| g_j \|_E^2 (\xi_j^2 - \lambda_j) \right)^{p/2} \leq d \left( \sum_{j} \mathbb{E} \| g_j \|_E^4 \lambda_j^2 \right)^{p/4},
\]
where \( d > 0 \) is a certain constant (obtained from \( \mathbb{E} \xi_j^4 = 3 \lambda_j^2 \)). Altogether we have proved
\[
\mathbb{E} \left( \int f \, d\xi \right)^p \leq C \| f \|_{p,2}^p + dC \left[ \sum_{1 \leq i,k} x_{ik} \| f_{ik} \|_E^4 (P_{ik} \rho(1_{t_{k+1}} - t_{k+1}))^2 \right]^{p/4}.
\]
Now the second term on the right side of this inequality goes to 0 if \( \sup_{k} (t_{k+1} - t_k) \) goes to 0, and the asserted inequality follows.

That this inequality is also sufficient for \( E \) to be 2-smoothable is proved similarly as the corresponding assertion of Theorem 1.2 (see [4]).

If \( E \) is not 2-smoothable, then we define
\[ \mathbb{W}_{p,\xi} := \text{the space of all } f \in \mathbb{W}_{p,2} \text{ such that there exists a sequence } (g_n) \text{ in } E \text{ with } \lim g_n = f \text{ (in } \mathbb{W}_{p,2} \text{) and such that} \]
\[ (\int g_n d\xi)_n \rightarrow \text{ is a Cauchy sequence in } \mathbb{W}_{p}(\mathcal{G}, E, p; E). \]
Again, \( \mathbb{W}_{p,\xi} \) is a Banach space with respect to the norm \( \| f \|_{p,\xi} \) given by
\[ \| f \|_{p,\xi} := \| f \|_{p,2} + (\mathbb{E} \left( \int f \, d\xi \right)^p)^{1/p}. \]
§2 - Stochastic integral equations

From now on we will assume that the variance measure \( \sigma \) of the Gaussian random field \( \xi \) is of the form \( \sigma = \lambda \varphi \) where \( \varphi \) is a \( \mathcal{F}_t \)-finite kernel from \( \mathbb{R}_+ \) to \( Z \), i.e. for \( s < t \) and \( \mathcal{C} \in \mathcal{F}_t \) we have

\[
\sigma(\mathcal{I}_s \times \mathcal{C}) = \int_s^t \varphi(r)(\mathcal{C})dr.
\]

We will consider stochastic integral equations of the form

\[
X_t = X_0 + \int_0^t g(r, X_r)dr + \int_0^t \int_Z G(r, z, X_r) \xi(dr, dz),
\]

where

(a) the initial condition \( X_0 \) is an \( E \)-valued \( \mathcal{F}_0 \)-measurable random vector with \( \mathbb{E}\|X_0\|^2 < \infty \).

(b) the drift function \( g: \mathbb{R}_+ \times E \to E \) is a continuous function with the following two additional properties:

\[
(L_g) \text{ Lipschitz continuity: There exists an increasing function } c_1: \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that for all } t \geq 0 \text{ and all } x, y \in E \\
\quad \|g(t, x) - g(t, y)\| \leq c_1(t)\|x - y\|. \]

\[
(B_g) \text{ Boundedness property: There exists an increasing function } c_2: \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that for all } t \geq 0 \\
\quad \|g(t, 0)\|^2 \leq c_2(t). \]

(c) the diffusion function \( G: \mathbb{R}_+ \times Z \times E \to E \) is a continuous function on \( \mathbb{R}_+ \times E \) for every fixed \( z \in Z \) and measurable on \( \mathbb{R}_+ \times \mathcal{F}_t \) and has the following further properties:

\[
(I_G) \text{ Integrability property: For every } t \geq 0 \text{ let } \mathbb{W}^2_t \text{ denote the Banach space of all predictable processes } X: [0, t] \times \Omega \to E \\
\quad \text{ such that } \|X\|_t := \sup_{s \leq t} (\mathbb{E}\|X_s\|^2)^{1/2} < \infty. \text{ For every } X \in \mathbb{W}^2_t \\
\quad \text{ let } G(X) \text{ denote the } E \text{-valued function on } \mathbb{R}_+ \times Z \times \Omega \text{ defined by } \\
\quad G(X)(s, z, \omega) := G(s, z, X_s(\omega)) \text{ for } s \leq t \text{ and } G(X)(s, z, \omega) := 0 \text{ for } s > t. \text{ } G(X) \text{ is necessarily predictable, but we assume in addition } \\
\quad G(X) \in \mathbb{W}^2_t \text{ for all } X \in \mathbb{W}^2_t \text{ (for all } t \geq 0). \]

\[
(L_G) \text{ Lipschitz continuity: There exists an increasing function } D: \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that for all } t \geq 0 \text{ and all } X, Y \in \mathbb{W}^2_t \\
\quad \mathbb{E}\|\int_0^t \int_Z [G(X) - G(Y)] d\xi\|^2 \leq D(t) \int_0^t \mathbb{E}\|X_r - Y_r\|^2 dr. \]

Under these assumptions we will prove that the stochastic integral equation (I) has a unique solution. But first we will look
for conditions which ensure that a function $G$ has properties $(I_G)$ and $(L_G)$, since these conditions are of course difficult to verify on a general Banach space. Indeed, the next result indicates, that the richness of the class of all diffusion functions (i.e. functions with properties $(I_G)$ and $(L_G)$) depends on the Banach space geometry.

2.1 Proposition. Assume that a function $G: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ factorizes through a 2-smoothable Banach space $F$ in the following sense: There exists a $T \in L(F, \mathbb{E})$ and a function $H: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{E} \to F$ which is jointly measurable and continuous on $\mathbb{R}_+ \times \mathbb{E}$, such that $G = T \circ H$ and such that $H$ has the properties:

$(I_H)$ There is a function $\varphi: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ such that

(i) for all $t > 0$, $z \in \mathbb{Z}$ and $x, y \in \mathbb{E}$

$$\|H(t, z, x) - H(t, z, y)\| \leq \varphi(t, z)\|x - y\|,$$

(ii) $\varphi(t, \cdot) \in L_2(\mathbb{Z}, \mathbb{E})$ for all $t > 0$,

(iii) the function $t \mapsto \varphi(t) := \int \varphi(t, z)^2 \varphi_t(dz)$ is locally bounded.

$(L_H)$ The function $t \mapsto \chi(t) := \int \|H(t, z, o)\|^2 \varphi_t(dz)$ is locally bounded.

Under these assumptions the function $G$ has properties $(I_G)$ and $(L_G)$.

Proof: It follows from the continuity and measurability properties of $H$ that for every $t > 0$ and every $x \in \mathbb{E}_t^2$ the function $(s, z, \omega) \mapsto H(s, z, X_s(\omega))$ is predictable. Since $F$ is 2-smoothable we get from $(I_H)$ and $(L_H)$

$$\mathbb{E}\left[\int_0^t \int_Z \|G(s, z, X_s)\|^2(ds, dz)\right] \leq C_s \|T\|^2 \mathbb{E}\left[\int_0^t \int_Z \|H(s, z, X_s)\|^2 \varphi_s(dz)ds\right]$$

$$\leq 2C_s \|T\|^2 \left\{\mathbb{E}\left[\int_0^t \int_Z \|H(s, z, X_s) - H(s, z, o)\|^2 \varphi_s(dz)ds\right]\right.$$ 

$$\left. + \mathbb{E}\left[\int_0^t \int_Z \|H(s, z, o)\|^2 \varphi_s(dz)ds\right]\right\}$$

$$\leq 2C_s \|T\|^2 \left[\sup_{s \in [t]} \mathbb{E}\|X_s\|^2 \int_0^t \varphi(s, z)^2 \varphi_s(dz)ds + \int_0^t \chi(s)ds\right] < \infty,$$

and $(I_G)$ follows essentially from this inequality. Condition $(L_G)$ follows from

$$\mathbb{E}\left[\int_0^t \int_Z [G(X) - G(Y)] d\xi \|^2\right] \leq C_s \|T\|^2 \mathbb{E}\left[\int_0^t \|H(X) - H(Y)\|^2 \varphi_s(dz)ds\right]$$

$$\leq C_s \|T\|^2 \int_0^t \|X_s - Y_s\|^2 \varphi_s(dz)ds = C_s \|T\|^2 \int_0^t \psi(s) \mathbb{E}\|X_s - Y_s\|^2 ds.$$
2.2 **Theorem.** Under the assumptions (a), (b), (c) on \(X_0, g\) and \(G\) resp., the stochastic integral equation (I) has a solution \((X_t)_{t \geq 0}\) which is almost surely continuous. If \((X'_t)_{t \geq 0}\) is another almost surely continuous solution of (I), then \(P[\sup_{s \leq t}\|X_s - X'_s\| = 0] = 1\) for all \(t \geq 0\).

**Proof:** Since the proof is similar to the classical case \(E = \mathbb{R}^d\) (see [8], ch.8 or [14], ch.5) we only present the main steps.

Let \(t > 0\) be fixed and define an operator \(S\) on \(\mathbb{H}_t^2\) by

\[
S_Y := X_0 + \int_0^T g(r,Y_r)dr + \int_0^T \int_G G(r,z,Y_r)\xi(dr,dz)
\]

for all \(s \in [0,t]\) and all \(Y \in \mathbb{H}_t^2\). By the assumptions on the functions \(g\) and \(G\) we get

\[
\mathbb{E}\|SY\|^2 \leq 3\mathbb{E}\|X_0\|^2 + 3\mathbb{E}\|\int_0^T g(r,Y_r)dr\|^2 + 3\mathbb{E}\|\int_0^T \int_G G(r,z,Y_r)\xi(dr,dz)\|^2
\]

\[
\leq 3\mathbb{E}\|X_0\|^2 + 6\mathbb{E}\|\int_0^T [g(r,Y_r) - g(r,o)]dr\|^2 + 6\mathbb{E}\|\int_0^T g(r,o)dr\|^2
\]

\[
+ 6\mathbb{E}\|\int_0^T \int_G [G(r,z,Y_r) - G(r,z,o)]\xi(dr,dz)\|^2 + 6\mathbb{E}\|\int_0^T G(r,z,o)d\xi\|^2
\]

\[
\leq 3\mathbb{E}\|X_0\|^2 + 6s_1(t)\int_0^T \mathbb{E}\|Y_r\|^2dr + 6s_2(t)
\]

\[
+ 6D(t)\int_0^T \mathbb{E}\|Y_r\|^2dr + 6\mathbb{E}\|\int_0^T G(r,z,o)d\xi\|^2 < \infty,
\]

and one obtains \(SY \in \mathbb{H}_t^2\), i.e. \(S\) maps \(\mathbb{H}_t^2\) into \(\mathbb{H}_t^2\).

For \(Y, Z \in \mathbb{H}_t^2\) we get for all \(s \in [0,t]\)

\[
\mathbb{E}\||SY - SZ||^2 \leq 2\mathbb{E}\|\int_0^T [g(r,Y_r) - g(r,Z_r)]dr\|^2
\]

\[
+ 2\mathbb{E}\|\int_0^T \int_G [G(r,z,Y_r) - G(r,z,Z_r)]\xi(dr,dz)\|^2
\]

\[
\leq K\int_0^T \mathbb{E}\|Y_r - Z_r\|^2dr \quad \text{with} \quad K = 2tc_1(t) + 2D(t),
\]

and hence \(\|SY - SZ\|_t^2 \leq K\int_0^t \|Y - Z\|_t^2 ds\). Induction yields

\[
\|S^nY - S^nZ\|_t^2 \leq \frac{k^n}{n!} \|Y - Z\|_t^2 \quad \text{for all} \ n \geq 1.
\]

This inequality now implies for an arbitrary but fixed \(Y \in \mathbb{H}_t^2\)

\[
\sum_{n \geq 0} \|S^{n+1}Y - S^nY\|_t \leq e^{Kt}\left[\|SY - Y\|_t + \|S^nY - SY\|_t \right] < \infty.
\]

Therefore \((S^nY)_{n \geq 1}\) is a Cauchy sequence in \(\mathbb{H}_t^2\) and it is easy to see that the limit \(Z = \lim S^nY\) has the property \(SZ = Z\). It follows \(Z = SZ\) a.s. on \([0,t]\). Now we put \(X_s := SZ_s\) for every \(s \in [0,t]\). Then
19

\[ (X_s)_{0 \leq s \leq t} \text{ is a.s. continuous and} \]

\[ \mathbb{E} \left[ \sup_{s \leq t} \| X_s - S_X s \|^2 \right] = \mathbb{E} \left[ \sup_{s \leq t} \| S_Z s - S(S_Z) s \|^2 \right] \]

\[ \leq 2t \mathbb{E} \left[ \int_0^t \| g(s, Z_s) - g(s, S_Z s) \|^2 ds + 2 \mathbb{E} \left[ \sup_{s \leq t} \left[ \int_0^t \| G(Z) - G(S_Z) \| d\hat{\xi} \right]^2 \right] \right] \]

\[ \leq 2t^2 c_1(t) \| Z - S_Z \|^2 t + 2 \mathbb{E} \left[ \int_0^t \| G(Z) - G(S_Z) \| d\hat{\xi} \right]^2 \]

\[ \leq K(t) \| Z - S_Z \|^2 t = 0 \text{, where } K(t) = 2t^2 c_1(t) + 2t D(t) \]

It follows that \((X_s)_{0 \leq s \leq t}\) is an a.s. continuous solution of (I).

Since \(t \to 0\) was arbitrary it follows that there is an a.s. continuous solution \((X_s)_{s > 0}\) of (I).

The uniqueness assertion can now be proved in the same way as in [14] (Corollary 5.1.2).

In the quoted proof of the uniqueness assertion in theorem 2.2 the following Gronwall lemma is used, which will frequently be applied in the sequel (cf. [8] for a proof).

2.3 Lemma. Let \(f: [0, t] \to \mathbb{R}_+\) and \(g: [0, t] \to \mathbb{R}_+\) be functions such that \(f\) is \((\lambda)-\)integrable and \(g\) is increasing. If there is a constant \(C > 0\), such that for all \(s \in [0, t]\)

\[ f(s) \leq C \int_0^s f(r) dr + g(s) \], then \( f(s) \leq g(s) e^{Cs} \) for all \(s \in [0, t]\).

Later on we need some slightly stronger conditions on the diffusion function \(G\) than \((I_G)\) and \((L_G)\). For \(p > 2\) we define:

\((I_{G_p})\) \(p\)-integrability property: For every \(t > 0\) let \(\mathbb{H}_{t}^{p}\) denote the Banach space of all predictable processes \(X: [0, t] \times \Omega \to \mathbb{E}\) with \("X"_{t,p} := \sup_{s \leq \tau} (\mathbb{E}[\| X(s) \|^p]^{1/p}) \). Then we assume \(G(X)\in \mathbb{H}_{t}^{p}\) for every \(X \in \mathbb{H}_{t}^{p}\) (for all \(t > 0\)).

\((L_{G_p})\) \(p\)-Lipschitz continuity: There exists an increasing function \(D_1: \mathbb{R}_+ \to \mathbb{R}_+\) (depending on \(p\)) such that for all \(t > 0\) and all \(X, Y \in \mathbb{H}_{t}^{p}\):

\[ \mathbb{E} \left[ \int_0^t \int_Z [G(X) - G(Y)] d\hat{\xi} \right]^p \leq D_1(t) \mathbb{E} \left[ \int_0^t \| X_r - Y_r \|^2 dr \right]^{p/2} . \]

\((B_{G_p})\) \(p\)-boundedness property: There exists an increasing function \(D_2: \mathbb{R}_+ \to \mathbb{R}_+\) such that for all \(0 \leq s \leq t\)

\[ \mathbb{E} \left[ \int_0^t \int_Z G(r, z, o) \xi(dr, dz) \right]^p \leq D_2(t)(t-s)^{p/2} . \]

2.4 Proposition. Suppose that \(G\) factorizes through a 2-smoothable Banach space \(F\) in the sense of prop.2.1, i.e. \(G = T \cdot H\), where \(H\) has
the properties \((L_H^0)\) and \((B_H^0)\). Then \(G\) has the properties \((I_G^p)\), \((L_G^p)\) and \((B_G^p)\) for all \(p \in [2,4]\).

Proof: Properties \((I_G^p)\) and \((B_G^p)\) follow from the inequality
\[
\mathbb{E}\|\int_0^t \int_Z G(r,z,X_r) \xi(dr,dz)\|^p \leq c_{s,p} \|T\|^p \mathbb{E}\left[\left\|\int_0^t H(r,z,X_r) \eta(r)dr\right\|^p/2\right],
\]
which is obtained from theorem 1.4. Property \((L_G^p)\) is also a consequence of theorem 1.4:
\[
\mathbb{E}\left[\int_0^t \int_Z [G(X) - G(Y)] d\xi\right]^p \leq c_{s,p} \|T\|^p \mathbb{E}\left[\int_0^t \left\|H(X) - H(Y)\right\|^2 d\beta\right]^{p/2}
\]

with \(K(t) = c_{s,p} \|T\|^p \sup_{s \leq t} \psi(s)^{p/2}\).

The next lemma compares the solutions of two different integral equations. A first application will be a result on the convergence of a sequence of diffusion processes.

2.5 Lemma. Let \(g': \mathbb{R}_+ \times E \rightarrow E\) and \(G': \mathbb{R}_+ \times Z \times E \rightarrow E\) be functions with the same properties - \((L_g)\), \((B_g)\) and \((I_g^p)\), \((L_g^p)\), \((B_g^p)\) resp. - as \(g\) and \(G\). If \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) denote the solutions of the stochastic integral equations \((I)\) and
\[
(I')\quad Y_t = Y_0 + \int_0^t g'(r,Y_r)dr + \int_0^t \int_Z G'(r,z,Y_r) \xi(dr,dz)\ 
\]
then for every stopping time \(T\) the following inequality holds:
\[
\mathbb{E}\sup_{s \leq t} \|X_{s\wedge T} - Y_{s\wedge T}\|^p \leq K(t) \left\{\mathbb{E}\|X_0 - Y_0\|^p + t^{p-1} \mathbb{E}\int_0^t \|g(r,Y_r) - g'(r,Y_r)\|^p dr\ 
+ \mathbb{E}\int_0^T \int_Z [G(r,z,Y_r) - G'(r,z,Y_r)] \xi(dr,dz)\|^p\right\}
\]
with \(K(t) = 5^{p-1} \exp \left[5^{p-1} t\psi_1(t)^{p/2} + 5^{p-1} t^{p/2}D_1(t)\right]\).

Proof: For every \(s \leq t\) we have a.s.
\[
\|X_{s\wedge T} - Y_{s\wedge T}\|^p \leq 5^{p-1} \left\{\|X_0 - Y_0\|^p + \int_0^{s\wedge T} \|g(r,X_r) - g'(r,Y_r)\| dr\|^p\ 
+ \int_0^{s\wedge T} \|g'(r,Y_r) - g(r,Y_r)\| dr+p \int_0^{s\wedge T} \int_Z [G(r,z,X_r) - G(r,z,Y_r)] \xi(dr,dz)\|^p\right\}
\]

Using the Lipschitz conditions \((L_g)\) and \((L_G^p)\), a straightforward application of lemma 2.3 then yields the asserted inequality.
2.6 Theorem. Let \((g_n)_{n \geq 0}\) be a sequence of drift functions (satisfying \((L_g)\) and \((B_g)\)) and let \((G_n)_{n \geq 0}\) be a sequence of diffusion functions (satisfying \((L_g)\) and \((B_g)\)). For all \(n \geq 0\) let \(X_n(t)\) be an a.s. continuous solution of

\[
\begin{align*}
(I_n) \quad X_n(t) &= x_0 + \int_0^t g_n(s, X_n(s)) \, ds + \int_0^t \int_Z G_n(s, z, X_n(s)) \, \xi(ds, dz),
\end{align*}
\]

where \(x_0 \in E\) is fixed. Then the paths of a subsequence of \((X_n)_{n \geq 1}\) converge a.s. uniformly on bounded intervals to the paths of \(X_0\), if the following two conditions hold for every bounded predictable process \(X\):

(i) \(\lim_{n \to \infty} E \int_0^t \|g_n(r, X_r) - g_0(r, X_r)\|^2 \, dr = 0\),

(ii) \(\lim_{n \to \infty} E \int_0^t \int_Z \|G_n(r, z, X_r) - G_0(r, z, X_r)\| \xi(dr, dz) \|^2 = 0\) for all \(t > 0\).

Conditions (i) and (ii) are valid under the following assumptions:

(1.1) \(\lim_{n \to \infty} g_n(t, x) = g_0(t, x)\) for all \(t > 0\), \(x \in E\).

(1.2) Uniform Lipschitz continuity of the drift functions: There exists an increasing function \(c_1: \mathbb{R}^+ \to \mathbb{R}^+\) such that

\[
\sup_{n \geq 0} \|g_n(t, x) - g_n(t, y)\|^2 \leq c_1(t) \|x - y\|^2
\]

for every \(t > 0\) and \(x, y \in E\).

(1.3) Uniform boundedness condition for the drift functions:

\[
\sup_{n \geq 0} \sup_{s \in \mathbb{R}} \|g_n(s, o)\|^2 =: c_2(t) < \infty
\]

for all \(t > 0\).

(2.1) There is a 2-smoothable Banach space \(F\) and an operator

\(T \in L(F, E)\)

such that \(G_n = T H_n\) for all \(n \geq 0\), where the functions

\(H_n: \mathbb{R}^+ \times Z \times E \to F\)

have the following properties:

(2.2) \(\lim_{n \to \infty} H_n(t, z, x) = H_0(t, z, x)\) for all \(t > 0\), \(z \in Z\) and \(x \in E\).

(2.3) Uniform Lipschitz continuity of the diffusion functions:

There exists a measurable function \(\varphi: \mathbb{R}^+ \times Z \to \mathbb{R}^+\) such that

\[
\varphi(t) := \int_Z \varphi(t, z)^2 \, \mu_t(dz) \quad (t > 0)
\]

defines a locally bounded function \(\varphi: \mathbb{R}^+ \to \mathbb{R}^+\) and such that

\[
\sup_{n \geq 0} \|H_n(t, z, x) - H_n(t, z, y)\| \leq \varphi(t, z) \|x - y\|
\]

for all \(t > 0\), \(z \in Z\) and \(x, y \in E\).

(2.4) Uniform boundedness condition for the diffusion functions:
The function $t \mapsto \chi(t) := \int \sup_{n \not= 0} \|H_n(t, z, o)\|^2 q_t(dz)$ is locally bounded.

**Proof:** For every $k > 0$ we put $T_k := \inf \{ t_0 : \|X_0(t)\| > k \}$. Let $\varepsilon > 0$ be given. For $k > 0$ large enough we have $P[T_k > t] \leq \varepsilon/2$ for a fixed $t_0$, and we get with the aid of lemma 2.5

$$
P\left[ \sup_{s \leq t} \|X_n(s) - X_0(s)\| > \varepsilon \right] \leq \varepsilon/2 + P\left[ T_k > t \right] \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for $n$ large enough. This proves that the sequence $(X_n)$ as a sequence of $C([0, t), \mathbb{E})$-valued random vectors converges in probability to $X_0$. Therefore a certain subsequence converges a.s. uniformly to $X_0$ on $[0, t]$. A diagonal argument then finally shows that there exists a subsequence $(X_{nk})$ which converges a.s. on every interval $[0, t]$ uniformly to $X_0$.

Now let us show that the conditions (1.1) to (2.4) imply the conditions (i) and (ii). Let $X$ be a fixed bounded predictable process.

**Proof of (i):** Since for every $\omega \in \Omega$ and every $s > 0$

$$
\lim_{n \to \infty} g_n(s, X_0(\omega)) = g_0(s, X_0(\omega)) \quad \text{by (1.1)}
$$

it is sufficient to prove that $\sup_{n > 0} \|g_n(s, X_0) - g_0(s, X_0)\|$ is bounded on every interval $[0, t]$. But this follows from (1.2) and (1.3) since

$$
\|g_n(s, X_0) - g_0(s, X_0)\| \leq \|g_n(s, X_0) - g_n(s, o)\| + \|g_n(s, o)\| + \|g_0(s, o)\| + \|g_0(s, X_0) - g_0(s, X_0)\|.
$$

**Proof of (ii):** We have

$$
E \left[ \int_0^t \int_Z \|G_n(s, z, X_0) - G_0(s, z, X_0)\|^2 (ds, dz) \right] \leq C_s T t^2 \left[ \int_0^t \int Z \|H_n(s, z, X_0) - H_0(s, z, X_0)\|^2 ds \right].
$$
Since

$$\|H_n(s,z,X_s) - H_0(s,z,X_s)\|^2 \leq 4 \left\{ \|H_n(s,z,X_s) - H_n(s,z,o)\|^2 + \|H_n(s,z,o)\|^2 + \|H_0(s,z,o)\|^2 + \|H_0(s,z,o) - H_0(s,z,X_s)\|^2 \right\}$$

$$\leq 8 \varphi(s,z)^2 \|X_s\|^2 + 8 \sup_{n \geq 0} \|H_n(s,z,o)\|^2 ,$$

it follows from (2.2), (2.3), (2.4) and Lebesgue's theorem that (ii) holds.

Theorem 2.6 has the following application, which shows that in many cases the solutions of stochastic integral equations of the form (I) can be approximated by finite-dimensional diffusions.

2.7 Corollary. Let $E$ be a 2-smoothable Banach space and suppose that there is a sequence $(P_n)_{n \geq 1}$ in $L(E)$ of operators of finite rank with $\sup_n \|P_n\| < \infty$ and $\lim_{n \to \infty} P_n(x) = x$ for all $x \in E$. Let $X$ be a solution of (I) with drift function $g$ and diffusion function $G$ (satisfying $(L_g)$ and $(B_G)$ and $(L_G^0)$, $(B_G^0)$ (see prop.2.1)). Define $g_n := P_n \cdot g$ and $G_n := P_n \cdot G$ for all $n \geq 1$ and denote by $X_n$ an a.s. continuous solution of

$$(I_n) \quad X_n(t) = P_n x_0 + \int_0^t g_n(s,X_n(s)) ds + \int_0^t \int \xi G_n(s,z,X_n(s)) \nu(dz,dr).$$

Then the assumptions of theorem 2.6 hold and consequently, there exists a subsequence $(X_{n_k})$ of $(X_n)$ converging a.s. uniformly on bounded time intervals to $X$.

§3 - Local properties of the solutions

Throughout this paragraph we will assume that $g: \mathbb{R}_+ \times E \to E$ is a drift function satisfying $(L_g)$ and $(B_g)$ and that $G: \mathbb{R}_+ \times \mathbb{R}^2 \times E \to E$ is a diffusion function satisfying $(I_G^4)$, $(L_G^4)$ and $(B_G^4)$.

For every $s \geq 0$ and $x \in E$ let now $X_{s,x} := (X_{s,x}(t))_{t \geq s}$ be an a.s. continuous solution of the stochastic integral equation

$$(I_{s,x}) \quad X_t = x + \int_s^t g(r,X_r) dr + \int_s^t \int \xi G(r,z,X_r) \nu(dz,dr).$$

One can prove (see [6] or [7]) that every process $X_{s,x}$ is a Markov process with transition probabilities given by

$$p(s,x)(t,y;u,A) := p[X_t,y(u) \in A] \quad (s \leq t \leq u, y \in E, A \in \mathfrak{F}(E)).$$

It follows that the family $\{X_{s,x} : s \geq 0, x \in E\}$ is a Markov family in the sense of [5] or [7](vol. II) and by a standard procedure one can show that there exists a Markov process

$$X = (\bar{O}, (\mathfrak{F}_t)_{t \geq s \geq t}, (X_t)_{t \geq s \geq t}, (\mathbb{P}_{s,x})_{s \geq 0, x \in E})$$
(cf. [7] for the notation and the proof of this fact). \( X \) is defined on the extended space \( \overline{\Omega} := \mathbb{R}^+ \times \mathbb{R}^n \) of elementary events, and the relation to the Markov family \((X_{s,x})_{s \geq 0, x \in E}\) is given by

\[
P_{s,x}[X_t \in A] = P\{X_{s,x}(t) \in A \} \quad (t \geq s, x \in E, A \in \mathcal{B}(E)).
\]

Moreover, we can also assume that the random field \( \xi \) is defined on \( \overline{\Omega} \): put \( \xi_t(c)(\omega) := \xi_{s+t}(c)(\omega) \) for \( \omega = (s,x,\omega) \). In the following the notation \( E_{s,x} \) is used for the expectation operator relative to the probability measure \( P_{s,x} \) on \( \overline{\Omega} \).

3.1 Proposition. \( X \) is a strong Markov process (see [7], vol.II).

Proof: We have to prove (cf. [5], Satz 5.9) that for every \( \omega \in \Omega, x,y \in E \) and \( f \in C^b(E) \) (real valued bounded continuous functions on \( E \)),

\[
\lim_{h \downarrow 0, y \to x} E_{s+h,y} f(X_t) = E_{s,x} f(X_t).
\]

Since \( E_{s+h,y} f(X_t) = E f(X_{s+h,y}(t)) \) and \( E_{s,x} f(X_t) = E f(X_{s,x}(t)) \), it is sufficient to prove

\[
\lim_{h \downarrow 0, y \to x} X_{s+h,x}(t) = X_{s,x}(t) \quad \text{in } L^2(\Omega, \mathcal{F}, P; E).
\]

We have

\[
E \|X_{s+h,y}(t) - X_{s,x}(t)\|^2 \leq 2 \left( I_1(h,y) + I_2(h) \right)
\]

with

\[
I_1(h,y) = E \|X_{s+h,y}(t) - X_{s+h,x}(t)\|^2
\]

and

\[
I_2(h) = E \|X_{s+h,x}(t) - X_{s,x}(t)\|^2 = E \|X_{s+h,x}(t) - X_{s+h,x}(s+h)(t)\|^2
\]

since \( X_{s,x}(t) = X_{s+h,x}(s+h)(t) \) p-a.s. for \( t \geq s+h \) by the uniqueness assertion of theorem 2.2. Now we can apply lemma 2.5 for \( I_1(h,y) \) and \( I_2(h) \): there exists a constant \( C(t) \) such that

\[
I_1(h,y) \leq C(t) \|x-y\|^2 \quad \text{and} \quad I_2(h) \leq C(t) E \|X_{s,x}(s+h) - x\|^4
\]

A further application of lemma 2.5 (with \( g'=0, G'=0 \)) yields that there exists a constant \( K \) (depending on \( s \) and \( x \)) such that

\[
E \|X_{s,x}(s+h) - x\|^4 \leq K h^2.
\]

Altogether we have obtained

\[
E \|X_{s+h,y}(t) - X_{s,x}(t)\|^2 \leq 2C(t)(\|x-y\|^2 + K^{1/2} h),
\]

and hence

\[
\lim_{h \downarrow 0, y \to x} X_{s+h,y}(t) = X_{s,x}(t) \quad \text{in } L^2(\Omega, \mathcal{F}, P; E).
\]

From now on let us assume that the functions \( g \) and \( G \) have the following additional properties:
(Ug) The drift function g is uniformly continuous on bounded subsets of \( \mathbb{R}^+ \times E \).

(Ug) For every \( t \geq 0, \; x \in E \) and \( x' \in E' \) the function \( z \mapsto G(t,z,x)^{\otimes^2}(x') \) := \( \langle G(t,z,x), x' \rangle G(t,z,x) \) belongs to \( L_2(\mathbb{Z}, \mathbb{P}_t; E) \), the function \( E' \ni x' \mapsto \int G(t,z,x)^{\otimes^2} \varphi_t(dz)(x') := \int G(t,z,x)^{\otimes^2}(x') \varphi_t(dz) \) belongs to \( E \otimes E \) (\( \varepsilon \)-tensor product, subspace of \( L(E', E'') \)), and the function

\[
R^+ \times E \ni (t,x) \mapsto \int G(t,z,x)^{\otimes^2} \varphi_t(dz) \in E \otimes E
\]

is uniformly continuous on bounded subsets of \( \mathbb{R}^+ \times E \).

3.2 Theorem. The Markov process \( X \) associated to the solutions of the stochastic integral equations \( (I_{s,x}) \) (\( s \geq 0, \; x \in E \)) has the following properties:

1. \( \lim_{h \to 0} h^{-2} \mathbb{E}_{t,x} \left[ \sup_{0 \leq r \leq h} \| X_{t+r} - x \|^4 \right] < \infty \),

2. \( \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} [X_{t+h} - x] = g(t,x) \) (convergence in \( E \)),

3. \( \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} [(X_{t+h} - x)^{\otimes^2}] = \int G(t,z,x)^{\otimes^2} \varphi_t(dz) \) (in \( E \otimes E \)),

where the convergence is uniform in \( (t,x) \) on bounded subsets of \( \mathbb{R}^+ \times E \).

Proof: 1. An application of lemma 2.5 (with \( g', G' = 0 \)) yields the following inequality on \( [0,u] \times kB \) (\( u > 0, \; k > 0, \; B = \) unit ball of \( E \)):

\[
\mathbb{E}_{t,x} \left[ \sup_{0 \leq r \leq h} \| X_{t+r} - x \|^4 \right] = \mathbb{E} \left[ \sup_{0 \leq r \leq h} \| X_{t,x(t+r)} - x \|^4 \right] \leq K(u) \left[ h^3 \int_t^{t+h} \| g(r,x) \|^4 dr + \mathbb{E} \left\| \int_t^{t+h} \int \| G(r,z,x) \| d\xi(dz,dr) \right\|^4 \right] \leq 8K(u) \left[ h^3 \left( \int_t^{t+h} \| g(r,x) - g(r,o) \|^4 dr \right) + \mathbb{E} \left\| \int_t^{t+h} \| g(r,o) \|^4 dr \right\|^4 + \mathbb{E} \left\| \int_t^{t+h} \| G(r,z,o) \| d\xi \right\|^4 \right] \leq 8K(u) \left[ h^4 (c_1(u) + c_2(u)) + h^2 (k^4 D_1(u) + D_2(u)) \right],
\]

and hence

\[
\lim_{h \to 0} h^{-2} \mathbb{E}_{t,x} \left[ \sup_{0 \leq r \leq h} \| X_{t+r} - x \|^4 \right] \leq 8K(u)(k^4 D_1(u) + D_2(u)) < \infty.
\]

2. For \( (t,x) \in [0,u] \times kB \) we get

\[
\| h^{-1} \mathbb{E}_{t,x} (X_{t+h} - x) - g(t,x) \|^2 = \| h^{-1} \mathbb{E} (X_{t,x(t+h)} - x) - g(t,x) \|^2
\]
and it follows from (1.) and (Ug) that the convergence in (2.) holds uniformly on $[0, u] \times kB$.

3. Since the operators occurring in (3.) are positive quadratic forms it is sufficient to prove

$$\lim_{h \to 0} h^{-1} \mathbb{E} \langle X_t, x(t+h)-x, x' \rangle^2 = \int_Z \langle G(t, z, x), x' \rangle^2 \phi_t(dz)$$

uniformly on $[0, u] \times kB \times B'$ ($B' := $ unit ball in $E'$) for all $u \geq 0$, $k > 0$. For an arbitrary $x \in B'$ we obtain

$$h^{-1} \mathbb{E} \langle X_t, x(t+h)-x, x' \rangle^2 - \int_Z \langle G(t, z, x), x' \rangle^2 \phi_t(dz)$$

$$= h^{-1} \mathbb{E} \left[ \int_t^{t+h} \langle g(s, X_t, x(s)), x' \rangle ds + \int_t^{t+h} \int_Z \langle G(s, z, X_t, x(s)), x' \rangle \xi(ds, dz) \right]^2$$

$$- \int_Z \langle G(t, z, x), x' \rangle^2 \phi_t(dz)$$

$$= A_h + 2B_h + C_h$$

with

$$A_h := h^{-1} \mathbb{E} \left[ \int_t^{t+h} \langle g(s, X_t, x(s)), x' \rangle ds \right]^2,$$

$$B_h := h^{-1} \mathbb{E} \left( \int_t^{t+h} \langle g(s, X_t, x(s)), x' \rangle ds \right) \left( \int_t^{t+h} \int_Z \langle G(s, z, X_t, x(s)), x' \rangle \xi(ds, dz) \right)$$

and

$$C_h := h^{-1} \mathbb{E} \left[ \int_t^{t+h} \int_Z \langle G(s, z, X_t, x(s)), x' \rangle \xi(ds, dz) \right]^2 - \int_Z \langle G(t, z, x), x' \rangle^2 \phi_t(dz).$$

We will prove now

(i) $\lim_{h \to 0} A_h = 0$, (ii) $\lim_{h \to 0} B_h = 0$, (iii) $\lim_{h \to 0} C_h = 0$

uniformly on $[0, u] \times kB \times B'$.

(i): $A_h \leq \mathbb{E} \left[ \int_t^{t+h} \langle g(s, X_t, x(s)), x' \rangle^2 ds \right] \leq \mathbb{E} \left[ \int_t^{t+h} \| g(s, X_t, x(s)) \|^2 ds \right]$

$$\leq 2h c_1(u) \sup_{t \leq s \leq t+h} \mathbb{E} \| X_t, x(s) - x \|^2 + 2 \int_t^{t+h} \| g(s, x) \|^2 ds.$$
and this implies \( \lim_{h \to 0} A_h = 0 \) uniformly on \([0, u] \times kB \times B'\).

(ii): We have

\[
E_h^2 \leq \left[ h^{-1} \mathbb{E} \left( \int_t^{t+h} g(s, X_t, x(s)) \, dx \right)^2 \right] \cdot \left[ h^{-1} \mathbb{E} \left( \int_t^{t+h} \mathcal{G}(X_t, x(s), x') \, dx' \right)^2 \right] 
\]

\[
= A_h \cdot \left[ h^{-1} \mathbb{E} \left( \int_t^{t+h} \mathcal{G}(s, X_t, x(s), x') \, dx' \right)^2 \right].
\]

Therefore (ii) follows, if (iii) is proved.

(iii): We have \( C_h \leq C_{1,h} + C_{2,h} \) with

\[
C_{1,h} = h^{-1} \mathbb{E} \left( \int_t^{t+h} \mathcal{G}(s, X_t, x(s)) \, dx \right)^2 - \left( \int_t^{t+h} \mathcal{G}(s, z, x(s), x') \, dx' \right)^2 
\]

and

\[
C_{2,h} = h^{-1} \mathbb{E} \left( \int_t^{t+h} \mathcal{G}(s, z, x(s), x') \, dx' \right)^2 .
\]

By property \((U_0)\) the last inequality implies \( \lim_{h \to 0} C_{2,h} = 0 \) uniformly on \([0, u] \times kB \times B'\). For \( C_{1,h} \) we get

\[
C_{1,h} \leq \left( h^{-1} \mathbb{E} \left( \int_t^{t+h} \mathcal{G}(s, z, X_t, x(s)) - \mathcal{G}(s, z, x) \right)^2 \right) \cdot \left( h^{-1} \mathbb{E} \left( \int_t^{t+h} \mathcal{G}(s, z, X_t, x(s)) + \mathcal{G}(s, z, x) \right)^2 \right)
\]

and it follows from \((U_0^4)\) and \((E_0^4)\) that \( \lim_{h \to 0} C_{1,h} = 0 \) uniformly on \([0, u] \times kB \times B'\).

From theorem 3.2 several nice local properties of \( X \) can be derived. First we will prove a result on the infinitesimal generator.

3.3 Proposition. Let \( X \) be a Markov process on \( E \) with the following properties:

1. For every \( u \to 0 \) and \( k \to 0 \) there exists a constant \( K(u, k) \) such that

\[
\lim_{h \to 0} h^{-2} \mathbb{E}_{t, x} [X_{t+h} - x]^4 \leq K(u, k) \quad \text{for all } (t, x) \in [0, u] \times kB .
\]

2. For every \( t \to 0 \) and \( x \in E \)

\[
\lim_{h \to 0} h^{-1} \mathbb{E}_{t, x} [X_{t+h} - x] =: b(t, x)
\]

exists in \( E \) and the convergence is uniform on \([0, u] \times kB \) for every \( u \to 0 \), \( k \to 0 \).

3. For every \( t \to 0 \) and \( x \in E \)
\[
\lim_{h \to 0} h^{-1} \mathbb{E}_{t, x}(X_{t+h} - x)^2 =: Q(t, x)
\]
even exists in \(L_{\mathcal{E}}(E)\) and the convergence is uniform on \([0, u] \times k\mathbb{B}\) for every \(u > 0\) and \(k > 0\).

Let \(F\) denote a second Banach space and let \(f: \mathbb{R}^+ \times E \to F\) be a function which is continuously differentiable with respect to the first variable and two times continuously differentiable with respect to the second variable. Let \(f_t: \mathbb{R}^+ \times E \to F\) and \(f_x: \mathbb{R}^+ \times E \to L(E, F)\) denote the first derivatives and let \(f_{xx}: \mathbb{R}^+ \times E \to L(E, L(E, F))\) denote the second derivative. The Banach space \(L(E; E, F)\) can be viewed as a subspace of \(L(E, L(E, F))\) and we assume in addition
a) \(f_{xx}\) is \(L(E; E, F)\)-valued and \(f_{xx}: \mathbb{R}^+ \times E \to L(E; E, F)\) is continuous,
b) \(f\) and the derivatives \(f_t, f_x, f_{xx}\) are bounded.

Under these assumptions the following limit relation holds:

\[
\lim_{h \to 0} h^{-1} \mathbb{E}_{t, x}\left[f(t+h, X_{t+h}) - f(t, x)\right] = f_t(t, x) + f_x(t, x)(b(t, x)) + \frac{1}{2} f_{xx}(t, x)(Q(t, x))
\]

for all \(t > 0\) and \(x \in E\). The convergence is uniform on bounded subsets of \(\mathbb{R}^+ \times E\), if the derivatives \(f_t, f_{xx}\) are uniformly continuous on bounded subsets of \(\mathbb{R}^+ \times E\).

**Proof:** By Taylor's formula we have

\[
\mathbb{E}_{t, x}\left[h^{-1}(f(t+h, X_{t+h}) - f(t, x))\right] = \mathbb{E}_{t, x}\left[h^{-1}(f(t+h, X_{t+h}) - f(t, X_{t+h}))\right] + \mathbb{E}_{t, x}\left[h^{-1}(f(t, X_{t+h}) - f(t, x))\right]
\]

\[
= \mathbb{E}_{t, x}\left[\int_0^1 f_t(t+rh, X_{t+h})\,dr\right] + \mathbb{E}_{t, x}\left[h^{-1} f_x(t, x)(X_{t+h} - x)\right]
\]

\[
+ \mathbb{E}_{t, x}\left[h^{-1}\left(\int_0^1 (1-r)f_{xx}(t, x+r(X_{t+h} - x))\,dr\right) (X_{t+h} - x)^2\right].
\]

(i) Since \(f_t\) is continuous in \((t, x)\) we can find for every \(\varepsilon > 0\) a \(\delta > 0\) such that \(|f_t(s, y) - f_t(t, x)| < \varepsilon\) if \(|s-t| < \delta\) and \(|x-y| < \delta\). Hence we get

\[
\mathbb{E}_{t, x}\left[\int_0^1 f_t(t+rh, X_{t+h})\,dr\right] - f_t(t, x)
\]

\[
\leq \int_0^1 \mathbb{E}_{t, x}\left[\int_0^1 f_t(t+rh, X_{t+h})\,dr\right] - f_t(t, x)\,dr
\]

\[
+ \int_0^1 \mathbb{E}_{t, x}\left[\int_0^1 f_{xx}(t+rh, X_{t+h})\,dr\right] - f_{xx}(t, x)\,dr
\]

\[
\leq 2 \|f_t\| \mathbb{E}_{t, x}[|X_{t+h} - x|] + \varepsilon \text{ for } h < \delta.
\]
(ii) From (2.) and the assumption on \( f_x \) we get
\[
\| \mathbb{E}_{t,x} \left[ h^{-1} f_x(t,x)(X_{t+h} - x) - f_x(t,x)(b(t,x)) \right] \|
\leq \| f_x(t,x) \| \| \mathbb{E}_{t,x} \left[ h^{-1}(X_{t+h} - x) \right] - b(t,x) \|.
\]

(iii) For a given \( \epsilon > 0 \) we choose \( \delta > 0 \) such that \( \| f_{xx}(t,y) - f_{xx}(t,x) \| < \epsilon \) if \( \| x - y \| \leq \delta \). Then we get from (3.) and the assumptions on \( f_{xx} \)
\[
\| \mathbb{E}_{t,x} \left[ \int_0^1 (1-r) f_{xx}(t,x+r(X_{t+h} - x)) dr \right] (h^{-1}(X_{t+h} - x)) \| 
\leq \frac{1}{2} \| f_{xx}(t,x) \| \| \mathbb{E}_{t,x} \left[ h^{-1}(X_{t+h} - x) \right] \|
\]
\[
\| \mathbb{E}_{t,x} \left[ \int_0^1 (1-r) f_{xx}(t,x+r(X_{t+h} - x)) - f_{xx}(t,x) \right] (h^{-1}(X_{t+h} - x)) \|
\leq \frac{1}{2} \| f_{xx}(t,x) \| \| \mathbb{E}_{t,x} \left[ h^{-1}(X_{t+h} - x) \right] \|
\]
\[
+ \| \mathbb{E}_{t,x} \left[ \int_0^1 (1-r) f_{xx}(t,x+r(X_{t+h} - x)) - f_{xx}(t,x) \right] (h^{-1}(X_{t+h} - x)) \|
\leq \frac{1}{2} \| f_{xx}(t,x) \| \| \mathbb{E}_{t,x} \left[ h^{-1}(X_{t+h} - x) \right] \|
\]
\[
+ \| \mathbb{E}_{t,x} \left[ \int_0^1 (1-r) f_{xx}(t,x+r(X_{t+h} - x)) - f_{xx}(t,x) \right] (h^{-1}(X_{t+h} - x)) \|
\]
\[
= \frac{1}{2} \| f_{xx}(t,x) \| \| \mathbb{E}_{t,x} \left[ h^{-1}(X_{t+h} - x) \right] \|
\]
\[
+ \| \mathbb{E}_{t,x} \left[ \int_0^1 (1-r) f_{xx}(t,x+r(X_{t+h} - x)) - f_{xx}(t,x) \right] (h^{-1}(X_{t+h} - x)) \|
\]
\[
+ \epsilon \| \mathbb{E}_{t,x} \left[ \int_0^1 (1-r) f_{xx}(t,x+r(X_{t+h} - x)) - f_{xx}(t,x) \right] (h^{-1}(X_{t+h} - x)) \|
\]

The inequalities in (i), (ii) and (iii) now imply the asserted convergence. This convergence is uniform on \([0,u] \times kB \) if the \( \delta > 0 \) in (i) and (iii) is independent of \((t,x)\) for \((t,x) \in [0,u] \times kB\), i.e. if \( f_t \) and \( f_{xx} \) are uniformly continuous on \([0,u] \times kB\).

The assumptions in proposition 3.3 about the boundedness of the derivatives of \( f \) are rather restrictive. Without these assumptions one can prove at least a local form of prop. 3.3.

3.4 Theorem. Let \( X \) be a continuous \( E \)-valued Markov process with the following properties:

(1.) For every \( t, r > 0 \) and \( \epsilon \in E \) let \( T \) be the stopping time
defined by \( T := \inf \{ s > t : \| X_s - x \| > r \} \). Then
\[
\lim_{h \to 0} h^{-1} \mathbb{P}_{t,x} [ T < t+h ] = 0
\]

(2.) \( \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} [ \mathbb{I}_{\| X_{t+h} - x \| \leq r} (X_{t+h} - x)^2 ] < \infty \) (for all \( t, r > 0, x \in E \)).

(3.) \( \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} [ \mathbb{I}_{\| X_{t+h} - x \| \leq \delta} (X_{t+h} - x) ] =: b(t,x) \)
exists for all \( t, r > 0, x \in E \).

(4.) \( \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} [ \mathbb{I}_{\| X_{t+h} - x \| \leq r} (X_{t+h} - x)^2 ] =: Q(t,x) \in E \delta E \).
exists in $E_{\mathcal{F}}$ for all $t > 0, r > 0, x \in E$.

Then for every function $f$ defined on a neighborhood $V$ of $(t, x)$ and with values in a second Banach space $F$ such that

a) the derivatives $f_t : V \rightarrow F, f_x : V \rightarrow L(E,F), f_{xx} : V \rightarrow L(E_{\mathcal{F}}, E,F)$ exist and are continuous,

b) $f_t, f_x$ and $f_{xx}$ are bounded on $V$,

the following limit relation holds:

$$\lim_{h \to 0} h^{-1} E_{t,x} \left[ f(t+h, X(t+h)_{\mathcal{F}}) - f(t,x) \right] = Af(t,x) \quad \text{with}$$

$$Af(t,x) := f_t(t,x) + f_x(t,x)(b(t,x)) + \frac{1}{2} f_{xx}(t,x)(Q(t,x)),$$

where $T = \inf \{ s \geq t : \| X_s - x \| \leq r \}$ with $r > 0$ small enough to have $(t, y) \in V$ if $\| y - x \| \leq r$.

Remark: The Markov process $X$ constructed from the solutions of the stochastic integral equations (It) obviously satisfies the conditions of the theorem with $b(t,x) := g(t,x)$ and $Q(t,x) := \int_0^2 G(t,x) \delta^2 d\mu_t$. This is a consequence of theorem 3.2.

Proof: We choose $u > t$ and $r > 0$ small enough to have $(s, y) \in V$ if $s \in [t, u]$ and $\| y - x \| \leq r$. Then

$$h^{-1} E_{t,x} \left[ f(t+h, X(t+h)_{\mathcal{F}}) - f(t,x) \right] = Af(t,x) = I_{1,h} + I_{2,h} + I_{3,h} + I_{4,h},$$

with

$$I_{1,h} = E_{t,x} \int_0^1 \left[ f_t(t+sh, X(t+h)_{\mathcal{F}}) - f_t(t,x) \right] ds,$$

$$I_{2,h} = E_{t,x} \left[ f_x(t,x)(h^{-1}(X(t+h)_{\mathcal{F}}) - x) - f_x(t,x)(b(t,x)) \right],$$

$$I_{3,h} = \frac{1}{2} E_{t,x} \left[ f_{xx}(t,x)((h^{-1}(X(t+h)_{\mathcal{F}}) - x) \delta^2 - f_{xx}(t,x)(Q(t,x)) \right],$$

$$I_{4,h} = E_{t,x} \left[ \int_0^1 (1-s)(f_{xx}(t,x+s(X(t+h)_{\mathcal{F}} - x)) - f_{xx}(t,x)) ds \right].$$

and one has to prove $\lim_{h \to 0} I_{4,h} = 0$ for $1 \leq j \leq 4$. We will only show $\lim_{h \to 0} I_{4,j} = 0$ since the other proofs are similar to the corresponding proofs in prop. 3.3.

By assumption a) on $f$ we can find for every $\varepsilon > 0$ a $\delta > 0$ such that $\| f_{xx}(t,y) - f_{xx}(t,x) \| < \varepsilon$ if $\| y - x \| < \delta$. Furthermore, there exists by assumption b) a constant $C > 0$ such that $\| f_{xx}(t,y) \| \leq C$ if $\| y - x \| \leq r$.

Put $S := \inf \{ s \geq t : \| X_s - x \| \geq \delta \}$. Then $S \leq T$ if $\delta \leq r$ and

$$\| I_{4,h} \| \leq C r^2 h^{-1} E_{t,x} \left[ S < t+h \right] + \frac{1}{2} E_{t,x} \left[ 1_{S \geq t+h} h^{-1} \| X(t+h)_{\mathcal{F}} - x \|^2 \right]$$
\[ \leq C r^2 h^{-4} \mathbb{P}_{t,x} [S < t+h] + \frac{1}{2} \int 1_{[\|X_{t+h} - x\| \leq r]} \cdot h^{-1} \|X_{t+h} - x\|^2 \cdot \mathbb{P}_{t,x} [S < t+h] + \frac{1}{2} \int 1_{[\|X_{t+h} - x\| \leq r]} \cdot h^{-1} \|X_{t+h} - x\|^2 \cdot \mathbb{P}_{t,x} [S < t+h]. \]

But this implies \( \lim_{h \to 0} \mathbb{P}_{t,x} = 0 \) by assumptions (1.) and (2.).

Now we will prove two local weak convergence properties of the Markov process \( X \) associated with the stochastic integral equations \( (I_t, x) \). Again we formulate the results more general to stress the crucial properties of \( X \) which are needed.

3.5 Theorem. Let \( E \) be a Banach space of type 2 (see e.g. [1]) and suppose that there is a uniformly bounded sequence \( (\mathbb{P}_n) \) in \( L(E) \) of operators of finite rank converging strongly to the identity on \( E \). Let \( X \) be a Markov process with the following properties valid for all \( t, \omega, x \in E \) and \( r > 0 \):

\[
\begin{align*}
(1.) \quad & \lim_{n \to \infty} \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} \left[ \frac{1}{2} \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \right] = 0, \\
(2.) \quad & \lim_{h \to 0} h^{-1} \mathbb{P}_{t,x} [\|X_{t+h} - x\| > r] = 0, \\
(3.) \quad & \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} \left[ \frac{1}{2} \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \right] = b(t,x) \in E \text{ exists}, \\
(4.) \quad & \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} \left[ \frac{1}{2} \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \right] = Q(t,x) \in E \text{ exists and } Q(t,x) \text{ is the covariance operator of a centered Gaussian measure } \mathcal{Q}(t,x) \text{ on } E. 
\end{align*}
\]

If \( M_h(t,x) := (X_{t+h} - x)(\mathbb{P}_{t,x}) \) denotes the distribution of \( X_{t+h} - x \), then

\[
\lim_{h \to 0} M_h(t,x) = \delta_{b(t,x)} * \mathcal{Q}(t,x) \quad \text{weakly.}
\]

Proof: We fix \( t \) and \( x \) and put \( \mathbb{P} := \mathbb{P}_{t,x} \). For a fixed \( n \geq 1 \) we put

\[
a := P_n(b(t,x)), \quad R := P_n \cdot Q(t,x), \quad \mathcal{Q} := P_n(\mathcal{Q}(t,x)), \quad \lambda^h := P_n(\lambda_h(t,x)).
\]

It follows from (2.), (3.), (4.) that

\[
\begin{align*}
(a) \quad & \lim_{h \to 0} \lambda^h(rB^c) = 0, \\
(b) \quad & \lim_{h \to 0} \int_{rB} y \lambda^h(dy) = a, \\
(c) \quad & \lim_{h \to 0} \int_{rB} y^2 \lambda^h(dy) = R
\end{align*}
\]

hold for every \( r > 0 \). We prove only (c) (the proofs of (a) and (b) are similar. Put \( K := \sup_n |P_n| \cdot \mathbb{P} \). Then it follows from (4.) that

\[
\lim_{h \to 0} \mathbb{E}_{t,x} \left[ \frac{1}{2} \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \cdot \mathbb{P}_{t,x} [S < t+h] \right] = \mathbb{E}_{t,x} \mathcal{Q}(t,x) = R.
\]
From (2.) we get
\[
\lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} \left\{ \left[ \sum_{k \in \mathbb{N}} \mathbb{P}(\|X_n(X_{t+h} - x)\| \leq r) \right] \right\} \leq \lim_{h \to 0} \int_{t,x} [\|X_{t+h} - x\| > r/k] \mathbb{P}_{n}(X_{t+h} - x) \delta_{a} \star \varphi \right\}
\]
and (c) follows. 

Since all the measures \( \lambda_{n} \) and \( \varphi \) are concentrated on the finite dimensional subspace \( P_{\mathbb{N}} \) of \( E \), the classical central limit theorem (cf. [1] or [2]) says that (a), (b) and (c) are equivalent to the weak convergence of the measures \( \lambda_{n}^{[h]} \) to the Gaussian measure \( \delta_{a} \star \varphi \).

To prove the weak convergence of the net \( \left( \mu_{h}(t,x) \right)_{h \to 0} \) it is sufficient to prove
\[
\lim_{h \to 0} \int f(y) \mu_{h}(t,x)[h^{*}](dy) = \int f(y) (\delta_{b}(t,x) \star \varphi(t,x))(dy)
\]
for every real bounded Lipschitz function \( f \) on \( E \) (see [9]).

For every \( h \to 0 \) let \( X_{n,1}, X_{n,2}, \ldots, X_{n,\mathbb{N}} \) be an independent sequence of \( E \)-valued random vectors all having the same distribution as \( X_{t+h} - x \). Let \( f \) be a fixed real valued bounded Lipschitz function. We have
\[
\left| \mathbb{E} f\left( \sum_{k \in \mathbb{N}} X_{h,k} \right) - \int f(y) (\delta_{b}(t,x) \star \varphi(t,x))(dy) \right| \leq I_{1}(n,h) + I_{2}(n,h) + I_{3}(n,h)
\]
with
\[
I_{1}(n,h) = \left| \mathbb{E} f\left( \sum_{k \in \mathbb{N}} X_{h,k} \right) - \mathbb{E} f\left( \sum_{k \in \mathbb{N}} P_{n} X_{h,k} \right) \right| \ ,
I_{2}(n,k) = \left| \mathbb{E} f\left( \sum_{k \in \mathbb{N}} P_{n} X_{h,k} \right) - \int f(P_{n} y)(\delta_{b}(t,x) \star \varphi(t,x))(dy) \right| \ ,
I_{3}(n) = \left| \int f(P_{n} y) - f(y) (\delta_{b}(t,x) \star \varphi(t,x))(dy) \right| .
\]
We have already proved \( \lim_{h \to 0} I_{2}(n,h) = 0 \) for every \( n > 1 \). By the assumptions on the sequence \( (P_{n}) \) we also have \( \lim_{n \to \infty} I_{3}(n,h) = 0 \). Since \( f \) is bounded and Lipschitz continuous we get for every \( r > 0 \)
\[
I_{1}(n,h) \leq 2 \left\| f \right\| \sum_{k \in \mathbb{N}} \mathbb{P}\left( \sup_{k}(X_{h,k} - P_{n} X_{h,k}) \right) > r \right\| + C \mathbb{E} \left[ \left\| \sup_{k}(X_{h,k} - P_{n} X_{h,k}) \right\| \leq r \right] \int \sum_{k} (X_{h,k} - P_{n} X_{h,k})
\]
\( (C = \text{Lipschitz constant}) \). With \( Z_{k} := \left\| \sup_{k}(X_{h,k} - P_{n} X_{h,k}) \right\| \leq r \)
\( (X_{h,k} - P_{n} X_{h,k}) \)
we obtain
\[
I_{1}(n,h) \leq J_{1}(n,h) + J_{2}(n,h) + J_{3}(n,h) \ ,
\]
where
Thus \( J_1(n,h) = 2 \mathbb{E} \left[ |(I-P_n)(X_{t+h}-X)|^2 \right] \), so \( J_2(n,h) \leq \mathbb{E} \left[ \sum_k (Z_k - \mathbb{E} Z_k)^2 \right] \) and \( J_3(n,h) = \mathbb{E} \left[ \sum_k Z_k \right] \).

By assumption (2.) we have \( \lim_{h \to 0} J_1(n,h) = 0 \). For \( J_3(n,h) \) we get

\[
J_3(n,h) \leq h^{-1} \mathbb{E} \left[ \left( \left\| (I-P_n)(X_{t+h}-X) \right\| \leq r \right) \left( \left\| (I-P_n)(X_{t+h}-X) \right\|^2 \right) \right] 
+ \mathbb{E} \left[ h^{-1} \mathbb{E} \left[ \left\| X_{t+h}-X \right\| \leq r(1+K)^{-4} \right] \right] 
+ \mathbb{E} \left[ \left\| X_{t+h}-X \right\| \leq r(1+K)^{-4} \right] \left( I-P_n \right)(X_{t+h}-X)
\]

and it follows from (2.) and (3.) that \( \lim_{h \to 0} J_3(n,h) = 0 \). Since \( E \) is of type 2 there exists a constant \( D > 0 \) such that

\[
\mathbb{E} \left[ \sum_k (Z_k - \mathbb{E} Z_k)^2 \right] \leq 4D \mathbb{E} \left[ \sum_k Z_k - \mathbb{E} Z_k \right]^2 \leq 4D \mathbb{E} \left[ \sum_k Z_k \right]^2
\]

and therefore

\[
\lim_{h \to 0} \mathbb{E} \left[ \left\| (I-P_n)(X_{t+h}-X) \right\|^2 \right] \leq 2 \mathbb{E} \left[ \int_t^{t+h} (I-P_n)g(s,X_t,x(s))ds \right]^2 + \mathbb{E} \left[ \int_t^{t+h} \int_Z (I-P_n)G(s,z,X_t,x)ds \right]^2
\]

and therefore

\[
\lim_{n \to \infty} \mathbb{E} \left[ \left\| (I-P_n)(X_{t+h}-X) \right\|^2 \right] \leq 2C \int_t^{t+h} \mathbb{E} \left[ \int_Z (I-P_n)G(s,z,X_t,x)ds \right]^2 \Psi_s(dz).
\]

Since \( \lim_{n \to \infty} P_n = I \) strongly, Lebesgue's theorem yields
\[ \lim_{n \to \infty} \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} \| (I-P_n)(X_{t+h}-x) \|^2 = 0. \]

3.7 **Theorem.** Let \( E \) be a Banach space (not necessarily of type 2) for which there exists a sequence \( (P_n) \) in \( L(E) \) as in theorem 3.5. Let \( X \) be a continuous \( E \)-valued Markov process with the following properties valid for all \( t \geq 0, x \in E \) and \( r > 0 \):

1. \[ \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x} \left[ \begin{array}{c} 1 \\ \| X_{t+h}-x \| \leq r \end{array} \right] (I-P_n)(X_{t+h}-x) \|^2 = 0, \]

2. \[ \lim_{h \to 0} h^{-1} \mathbb{P}_{t,x} [T < t+h] = 0, \]

and there exist \( u > t, c > 0 \) such that

3. \[ \sup_{t \leq s \leq u, \| y-x \| \leq c, h \leq u-t} h^{-1} \mathbb{E}_{s,y} \left[ \| X_{s+h}-y \| > r \right] < \infty, \]

4. \[ \sup_{t \leq s \leq u, \| y-x \| \leq c, h \leq u-t} h^{-2} \mathbb{E}_{s,y} \left[ \begin{array}{c} 1 \\ \| X_{s+h}-y \| \leq r \end{array} \right] \| X_{s+h}-y \|^4 < \infty, \]

5. \[ \lim_{h \to 0} h^{-1} \mathbb{E}_{s,y} \left[ \begin{array}{c} 1 \\ \| X_{s+h}-y \| \leq r \end{array} \right] (X_{s+h}-y) =: b(s,y) \]

exists uniformly on \([t,u] \times (cB+x)\) and \( b \) is continuous in \((t,x)\),

6. \[ \lim_{h \to 0} h^{-1} \mathbb{E}_{s,y} \left[ \begin{array}{c} 1 \\ \| X_{s+h}-y \| \leq r \end{array} \right] (X_{s+h}-y)^2 =: Q(s,y) \]

exists uniformly on \([t,u] \times (cB+x)\) in \( E \otimes E \), \( Q \) is continuous in \((t,x)\) and \( Q(t,x) \) is the covariance operator of a centered Gaussian measure \( \varrho(t,x) \) on \( E \).

Under all these assumptions

\[ \lim_{h \to 0} (h^{-1/2}(X_{t+h}-x))(\mathbb{P}_{t,x}) = \varrho(t,x) \] weakly

for all \( t \geq 0 \) and all \( x \in E \).

**Proof:** As in the proof of theorem 3.5 we first prove that for every fixed \( n \geq 1 \)

\[ \lim_{h \to 0} (h^{-1/2} P_n(X_{t+h}-x))(\mathbb{P}_{t,x}) = P_n(\varrho(t,x)) \] weakly

for all \( t \geq 0, x \in E \). Put \( m(h) := [h^{-1}] \). Then

\[ h^{-1/2}(P_n X_{t+h} - P_n x) = \sum_{1 \leq k \leq m(h)} Y_{hk} + Z_h \] with

\[ Y_{hk} = h^{-1/2}(P_n X_{t+kh} - P_n X_{t+(k-1)h}) \] and

\[ Z_h = h^{-1/2}(P_n X_{t+h} - P_n X_{t+[h^*]h^2}). \]
As in the proof of theorem 3.5 we put \( \mathcal{Q} = P_n(\mathcal{Q}(t,x)) \), \( \mathbb{P} = \mathbb{P}_t,x \) and \( R = P_{n.}^2.Q(t,x) \). We will now prove

\[ (\ast) \lim_{h \to 0} \left( \sum_{1 \leq k \leq m(h)} Y_{hk} \right)(\mathbb{P}) = \mathcal{Q} \quad \text{weakly} \]

It is not difficult to see that this is equivalent to

\[ \lim_{h \to 0} (h^{-1/2}(P_n X_{t+h} - P_n X))(\mathbb{P}) = \mathcal{Q} \quad \text{weakly} \]

Proof of (\( \ast \)): For \( k = 0, 1, \ldots, m(h) - 1 \) we put \( \mathcal{Q}_{h,k} := \mathcal{Q}^t_{t+kh^2} \). Then we have to prove the following assertions (see [13]):

(I) \( \lim_{h \to 0} \sum_{1 \leq k \leq m(h)} \mathbb{P}[\|Y_{hk}\| > r | \mathcal{Q}_{h,k-1}] = 0 \),

(II) \( \lim_{h \to 0} \sum_{1 \leq k \leq m(h)} \mathbb{E}[\mathbb{1}[\|Y_{hk}\| \leq r] Y_{hk} | \mathcal{Q}_{h,k-1}] = 0 \),

(III) \( \lim_{h \to 0} \sum_{1 \leq k \leq m(h)} \mathbb{E}[\mathbb{1}[\|Y_{hk}\| > r] Y_{hk}^2 | \mathcal{Q}_{h,k-1}] = R \)

in probability for every \( r > 0 \).

[Remark: The conditions (II) and (III) are simpler than those in [13] since we have to prove convergence towards a Gaussian measure, see [2], Kor. 3.11].

Proof of (I): We prove that the random variables on the left converge to 0 in the mean. We have the following chain of inequalities:

\[ \mathbb{E}\left[ \sum_{1 \leq k \leq m(h)} \mathbb{P}[\|Y_{hk}\| > r | \mathcal{Q}_{h,k-1}] \right] \leq \sum_{1 \leq k \leq m(h)} \mathbb{P}[T < t+(k-1)h^2] + \sum_{1 \leq k \leq m(h)} \mathbb{E}\left[ \mathbb{1}[T \geq t+(k-1)h^2] \mathbb{E}_{t+(k-1)h^2} x_{t+(k-1)h^2} \mathbb{E}_{t+(k-1)h^2} x_{t+(k-1)h^2} \mathbb{P}[\|Y_{hk}\| > r] \right] \leq h^{-1} \mathbb{P}[T < t+h] \]

\[ + \frac{1}{h^{1/2}} \sup_{t \leq s \leq t+h} \mathbb{P}[\|X_{s+h^2}-y\| > r/k] \]

\[ + (k/r)^4 h \sup_{t \leq s \leq t+h} \mathbb{P}[\|X_{s+h^2}-y\| \leq r/k] \mathbb{E}_{s,y}[\|X_{s+h^2}-y\|^4] \]

The assumptions (2.), (3.) and (4.) now imply
\[
\lim_{h \downarrow 0} \mathbb{E} \left\{ \sum_{1 \leq k \leq m(h)} \mathbb{E} \left[ \left| Y_{hk} \right| \left| g_{h,k-1} \right| \right] \right\} = 0
\]

and (I) is proved.

**Proof of (II):** We have

\[
\mathbb{E} \left\{ \sum_{1 \leq k \leq m(h)} \mathbb{E} \left[ \left| Y_{hk} \right| \left| g_{h,k-1} \right| \right] \right\} \leq \sum_{k} \mathbb{E} \left\{ \mathbb{E} \left[ \mathbb{E} \left[ \left| Y_{hk} \right| \left| g_{h,k-1} \right| \right] \right] \right\} \leq r^{-1} \mathbb{P} \left[ T < t + h \right]
\]

\[
+ \sum_{k} \mathbb{E} \left\{ \mathbb{E} \left[ \left| t + (k-1)h^2 \right| \left| \mathbb{E} \left[ (k-1)h^2, X_{t+(k-1)h^2}, \mathbb{E} \left[ \left| Y_{hk} \right| \left| g_{h,k-1} \right| \right] \right] \right] \right\} \leq r^{-1} \mathbb{P} \left[ T < t + h \right]
\]

\[
+ h^{-1} \sup_{s \neq k} \mathbb{E} \left[ \mathbb{E} \left[ \left| X_{t+(k-1)h^2} - y \right| \right] \right] \leq \mathbb{E} \left[ \left| X_{t+(k-1)h^2} - y \right| \right] \}
\]

\[
\leq r \cdot h^{-1} \mathbb{P} \left[ T < t + h \right] + A(h) + B(h)
\]

with

\[
A(h) \leq K \cdot h^{-3/2} \sup_{s \neq k} \left\{ \mathbb{E} \left[ \left| X_{t+(k-1)h^2} - y \right| \right] \right\}
\]

\[
\leq K \cdot h^{-1/2} \sup_{s \neq k} \left\{ \mathbb{E} \left[ \left| X_{t+(k-1)h^2} - y \right| \right] \right\}
\]

and

\[
B(h) \leq h^{-1} \sup_{s \neq k} \left\{ \mathbb{E} \left[ \left| X_{t+(k-1)h^2} - y \right| \right] \right\}
\]

and it follows from (2.) to (5.) that

\[
\lim_{h \downarrow 0} \mathbb{E} \left\{ \sum_{k} \mathbb{E} \left[ \left| Y_{hk} \right| \left| g_{h,k-1} \right| \right] \right\} = 0.
\]

**Proof of (III):** Again we prove convergence in the mean.

\[
\mathbb{E} \left\{ \sum_{k} \left( \mathbb{E} \left[ \left| Y_{hk} \right| \left| g_{h,k-1} \right| \right] - [h^{-1}]^{-1} \right) \right\}
\]

\[
\leq h^{-1} (r^2 + K^2 \left\{ Q(t,x) \right\}) \mathbb{P} \left[ T < t + h \right] + C(h) + D(h)
\]

with

\[
C(h) \leq \sum_{k} \mathbb{E} \left[ \left| t \mathbb{E} \left[ \left| Q(s_k, X_{t+k}) - Q(t,x) \right| \right] \right] \right\}
\]

\[
\text{(with } s_k := t+(k-1)h^2)\]

\[
D(h) \leq h K^2 \left\{ Q(t,x) \right\}
\]

\[
\text{and } h \cdot h^{-1} \mathbb{P} \left[ T < t + h \right] + A(h) + B(h)
\]

with
\[ D(h) = \sum_k \mathbb{E}\left[ \left( T - s_k \right) \left( X_{s_k} - x \right) \right] + h Q(t,x) \bigg| \ \text{and} \ \text{q} \ \text{where} \ \text{R}(s,y) := \mathbb{P}_n Q(s,y). \ \text{With the abbreviations} \ \text{y}_{s,y} := \mathbb{P}_n (X_{s+h^2} - y), \ X_{s,y} := X_{s+h^2} - y \ \text{and} \ r' := r/K \]

we obtain further on

\[ D(h) \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s+h^2} - y \right] (h^{-1/2} y_{s,y})^2 - h R(s,y) \]

\[ \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s,y} \right] (h^{-1/2} y_{s,y})^2 - h R(s,y) + h R(s,y) \]

\[ \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s,y} \right] ^2 \]

and

\[ D(h) \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s+h^2} - y \right] \]

\[ \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s+h^2} - y \right] \]

\[ \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s+h^2} - y \right] \]

\[ \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s+h^2} - y \right] \]

\[ \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s+h^2} - y \right] \]

\[ \leq h^{-1} \sup_{y-x \in c} \mathbb{E}_{s,y}\left[ X_{s+h^2} - y \right] \]

From property (4.) we get \( \lim_{h \to 0} D(h) \leq \epsilon^2 C \), where
\[ C = K^2 r^{-\gamma} \sup_{y \neq u \in C} h^{-2} \mathbb{E}_{s,y} \left[ \mathbb{1}_{\|X_{s+h} - y\| \leq r} \|X_{s+h} - y\|^4 \right] < \infty, \]

and together with (2.), (3.) and (6.) we have obtained

\[ \lim_{h \to 0} \mathbb{E} \left[ \sum_k \left( \mathbb{E} \left[ F_{h,k} \|Y_{h,k} \| \leq r \right] \phi_{h,k} \right) - \left[ h^{-1} \right]^{-1} \right] \leq \varepsilon^2 C. \]

Letting \( \varepsilon \to 0 \) we get property (III).

As in the proof of theorem 3.5 let now \( f : E \to \mathbb{R} \) be an arbitrarily bounded Lipschitz function. Then

\[ \mathbb{E} \left[ f\left( h^{-1/2}(X_{t+h} - x) \right) \right] - \int f(u) \phi(t,x)(du) \leq I_1(n,h) + I_2(n,h) + I_3(n) \]

with

\[ I_1(n,h) = \mathbb{E} \left[ f\left( h^{-1/2}(X_{t+h} - x) \right) - f\left( h^{-1/2}(P_n X_{t+h} - P_n x) \right) \right], \]

\[ I_2(n,h) = \mathbb{E} \left[ f\left( h^{-1/2}(P_n X_{t+h} - P_n x) \right) \right] - \int f(P_n u) \phi(t,x)(du), \]

\[ I_3(n) = \int f(P_n u) - f(u) \phi(t,x)(du). \]

We have already proved \( \lim_{h \to 0} I_2(n,h) = 0 \) for every \( n \geq 1 \) and it follows from the assumptions on the sequence \( (P_n) \) that \( \lim_{n \to \infty} I_3(n) = 0 \).

For the first term we get with \( r > 0 \),

\[ I_1(n,h) \leq 2 \| f \| \mathbb{P} \left[ \|X_{t+h} - x\| > r \right] \]

\[ + h^{-1} C \mathbb{E} \left[ \mathbb{1}_{\|X_{t+h} - x\| \leq r} \right] \left\| (I-P_n)(X_{t+h} - x) \right\|^2 \]

and hence by assumption (1.)

\[ \lim_{n \to \infty} \lim_{h \to 0} I_1(n,h) = 0. \]

This proves

\[ \lim_{h \to 0} \mathbb{E} \left[ f\left( h^{-1/2}(X_{t+h} - x) \right) \right] = \int f \phi(t,x) \]

for every bounded Lipschitz function and the asserted weak convergence follows (see [9]).

Theorem 3.7 is valid without assumptions on the geometry of the Banach space \( E \). Nevertheless, corollary 3.6 shows that the validity of property (1.) requires some restrictions on the geometry.

The fact that in contrast to theorem 3.5 the weak limit in theorem 3.7 does not depend on the drift term \( b(t,x) \) leads to the following martingale property.
3.8 Corollary. Let $X$ be a continuous strong Markov process on $E$ with the following properties valid for all $t \geq 0$, $x \in E$ and $r > 0$:

1. \[ \lim_{h \to 0} h^{-1} P_{t,x}[T < t+h] = 0 \quad (T = \inf\{s \geq t : \|X_s - x\| > r\}) , \]

2. \[ \lim_{h \to 0} h^{-1} \mathbb{E}_{t,x}[1_{\|X_{t+h} - x\| \leq r}] (X_{t+h} - x) = g(t, x) , \]

where $g : \mathbb{R}^+ \times E \to E$ is continuous and bounded on bounded subsets of $\mathbb{R}^+ \times E$.

3. \[ \lim_{h \to 0} (h^{-1/2}(X_{t+h} - x))(\mathbb{P}_{t,x}) = \mathcal{Q}(t, x) \text{ weakly} , \]

is a centered Gaussian measure.

Define $M_{t,x}(s) := X_{(t+s) \wedge T} - x - \int_t^{(t+s) \wedge T} g(u, X_u) du$ for all $t \geq 0$, $x \in E$, $r > 0$ and $T = \inf\{s \geq t : \|X_s - x\| > r\}$. Then $(M_{t,x}(s))_{s \geq 0}$ is a martingale relative to $(\Omega, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P}_{t,x})$ with the property

\[ \lim_{h \to 0} (h^{-1/2} M_{t,x}(h)) (\mathbb{P}_{t,x}) = \mathcal{Q}(t, x) \text{ weakly} . \]

**Proof:** Define functions $F : \mathbb{R}^+ \times E$ and $f : \mathbb{R}^+ \times E$ by

\[ F(s) := \mathbb{E}_{t,x} X_{(t+s) \wedge T} \text{ and } f(s) := \mathbb{E}_{t,x}[1_{T > t+s}] g(t+s, X_{(t+s) \wedge T}) \]

Then it follows from the strong Markov property and condition (2.) that $\frac{dF(s)}{ds} = f(s)$ for all $s \geq 0$. Hence $F(v) - F(u) = \int_u^v f(s) ds$ for $0 \leq u \leq v$ and the martingale property of $M_{t,x}$ can easily be derived from this relation. To prove the asserted weak convergence, let $f : E \to \mathbb{R}$ be an arbitrary bounded Lipschitz function. Then there is a constant $C(f)$ such that

\[ |\mathbb{E}_{t,x} f(h^{-1/2} M_{t,x}(h)) - \mathbb{E}_{t,x} f(h^{-1/2}(X_{(t+h) \wedge T} - x))| \leq C(f) \mathbb{E}_{t,x} \|h^{-1/2} \int_t^{(t+h) \wedge T} g(s, X_s) ds\| \]

Thus inequality implies the asserted weak convergence, since by (1.) we have also \[ \lim_{h \to 0} (h^{-1/2}(X_{(t+h) \wedge T} - x))(\mathbb{P}_{t,x}) = \mathcal{Q}(t, x) . \]

§4 - Approximation by Markov chains

As in the last paragraph we assume that the drift function $g : \mathbb{R}^+ \times E \to E$ and the diffusion function $G : \mathbb{R}^+ \times \mathbb{R} \times E \to E$ have the
properties \((L_g),(B_g)\) and \((I_G^4),(L_G^4),(B_G^4)\) resp. We will prove in this paragraph that the solutions of the stochastic integral equations \((I_{a,x})\) can also be obtained as limits of Markov chains.

For all \(0 \leq t \leq s\) and \(x \in \mathbb{E}\) we put

\[
Z_{t,x}(s) := x + \int_t^s g(r,x)dr + \int_t^s \int_Z G(r,z,x)\xi(dr,dz).
\]

\(Z_{t,x}\) is an \(\mathbb{E}\)-valued Gaussian process with independent increments whose distributions are given by

\[
Z_{t,x}(s)(\mathbb{P}) = \mathcal{N}(a(t,x)(S),\mathcal{Q}_t,x)\sim,\mathcal{Q}_t,x(s) , \text{ where}
\]

(i) \(a(t,x)(s) := \int_t^s g(r,x)dr \in \mathbb{E}\) and

(ii) \(\mathcal{Q}_t,x(s)\) is the centered Gaussian measure with covariance operator \(Q_{t,x}(s) := \int_t^s \int_Z G(r,x,x)Q \mathcal{Q}_s(dz)dr\).

Let \(\Pi\) denote the family of all sequences \((t_k)_{k \geq 0}\) in \(\mathbb{R}_+\) with \(t_0 = 0\) and \(t_k < t_{k+1} \neq \infty\) such that \(\{(t_k)_{k \geq 0}\} := \sup_{k \geq 0} (t_{k+1} - t_k) < \infty\).

For every \(\Delta = (t_k)_{k \geq 0} \in \Pi\) we define inductively a process

\[
X^\Delta : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow C(\mathbb{R}_+, \mathbb{E}) (= \text{continuous } \mathbb{E}\text{-valued functions on } \mathbb{R}_+)
\]

by

\[
X^\Delta(o) := x_o \text{ (where } x_o \text{ is a fixed element in } \mathbb{E}) \text{ and}
\]

\[
X^\Delta(t) := Z_{t_k,x}(t_k)(t) \text{ for } t \in [t_k, t_{k+1}[^k \geq 0).
\]

The process \(X^\Delta = (X^\Delta(t))_{t \geq 0}\) is not a Markov process, but \((X^\Delta(t_k))_{k \geq 0}\) is a Markov chain relative to \((\mathcal{F}_t)\). Let \(\varphi^\Delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) denote the function defined by \(\varphi^\Delta := \sum_{k \geq 0} t_k 1_{[t_k, t_{k+1}[} \cdot\)

Then the process \(X^\Delta\) can be written in the form:

\[
(1.4) \quad X^\Delta(t) = x_o + \int_0^t g(r, X^\Delta(r) \varphi^\Delta(r))dr + \int_0^t \int_Z G(r,z, X^\Delta \varphi^\Delta(r))\xi(dr,dz).
\]

For every \(t \geq 0\) let us denote by \(X^\Delta,t\) the process \(X^\Delta\) restricted to \([o,t]\). Then every \(X^\Delta,t\) is a random vector in the Banach space \(C([o,t], \mathbb{E})\). We will prove the following theorem:

4.1 Theorem. For every \(t \geq 0\) the net \((X^\Delta,t)_{\Delta \in \Pi}\) of \(C([o,t], \mathbb{E})\)-valued random vectors converges in probability, and there is a subsequence \((\Delta_n)\) such that \((X^{\Delta_n})\) converges \(P\)-a.s. uniformly on every interval \([o,t]\) to a solution of the stochastic integral equation:
\[(I) \quad X_t = x_0 + \int_0^t g(s,X_s)ds + \int_0^t \int Z G(s,z,X_s)\xi(ds,dz).\]

We prove the theorem by a sequence of lemmas.

4.2 Lemma. For every \(t \geq 0\) and every stopping time \(T\)
\[\mathbb{E} \sup_{s \leq T} |X^\Delta(s,T) - x_0|^4 \leq 64 t^2(D_2(t) + t^2c_1(t)^2) K(t)\]
with \(K(t) = \exp(64t^2(D_1(t) + t^2c_1(t)^2))\). \([c_2\text{ and } D_2\text{ here depend on } x_0^0]\)

**Proof:** \[\mathbb{E} \sup_{s \leq T}|X^\Delta(s,T) - x_0|^4 \leq 8 \mathbb{E} \sup_{s \leq T} \int_0^{s \wedge T} g(r,X^\Delta(r))dr|^4 + 8 \mathbb{E} \sup_{s \leq T} \int_0^{s \wedge T} G(r,z,X^\Delta(r))d\xi|^4\]
\[\leq 8 t^3 \int_0^T \mathbb{E} \sup_{r \leq s} |g(r,X^\Delta(r))|^4 dr + 8 \mathbb{E} \int_0^T \int Z G(r,z,X^\Delta(r))d\xi|^4\]
(since \(\int_0^T G(r,z,X^\Delta(r))d\xi|^4\) \(t \geq 0\) is a submartingale)
\[\leq 64 t^2 \int_0^T \mathbb{E} \sup_{r \leq s} |g(r,X^\Delta(r))|^4 dr + 64t^2 \int_0^T |g(r,x_0)|^4 dr\]
\[+ 64 \mathbb{E} \int_0^T \int Z [G(t,z,X^\Delta(t)) - G(r,z,x_0)] d\xi|^4 + 64 \mathbb{E} \int_0^T \int Z G(r,z,x_0)d\xi|^4\]
\[\leq 64t^2 c_1(t)^2 + D_1(t)) \int_0^T \mathbb{E} \sup_{t \leq r} |X^\Delta(r) - x_0|^4 dr + 64t^2(D_2(t) + t^2c_2(t)^2) K(t)\]
An application of lemma 2.3 now yields the asserted inequality.

4.3 Lemma. For every \(\Delta \in \mathcal{T}\) and \(k > 0\) let \(T^\Delta_k := \inf\{t \geq 0: \|X^\Delta(t) - x_0\|^4 > k\}\).

Then \( \lim_{k \to \infty} \sup_{\Delta \in \mathcal{T}} \mathbb{P}[T^\Delta_k < t] = 0 \) for every \(t \geq 0\).

**Proof:** We have
\[\mathbb{P} \left[ \sup_{s \leq T} \|X^\Delta(s) - x_0\|^4 \leq k^{-4} \right] \leq k^{-4} \mathbb{E} \left[ \sup_{s \leq T} \|X^\Delta(s \wedge T^\Delta_k) - x_0\|^4 \right]\]
\[\leq k^{-4} 64t^2(D_2(t) + t^2c_2(t)^2) K(t)\] by lemma 4.2. Since the right side of this inequality is independent of \(\Delta\), the assertion of the lemma follows.

4.4 Lemma. Define
\[\text{SX}^\Delta(t) := x_0 + \int_0^t g(r,X^\Delta(r))dr + \int_0^t \int Z G(r,z,X^\Delta(r))\xi(dr,dz).\]

Then for every stopping time \(T\) the following inequality holds
\[\mathbb{E} \left[ \sup_{s \leq T} \|X^\Delta(s \wedge T) - \text{SX}^\Delta(s \wedge T)\|^4 \right] \leq C(t) \mathbb{E} \left[ \sup_{s \leq T} \|X^\Delta(s \wedge T) - X^\Delta(s \wedge T)\|^4 \right]\]
with \(C(t) = 8t^2(t^2c_1(t)^2 + D_1(t))\).
Proof: We have
\[ \mathbb{E} \left[ \sup_{s \in \mathbb{T}} \left\| X^A(s_{T}) - X^A(s_{AT}) \right\|_4^4 \right] \]
\[ \leq 8 t^3 \mathbb{E} \int_0^t \| g(r, X^A(r_{T})) - g(r, x^A(r_{AT})) \|_4^4 dr + 8 \mathbb{E} \left[ \int_0^t \int_{\mathbb{Z}} \left( G(r, z, X^A(r_{T})) - G(r, z, x^A(r_{T})) \right) \xi \|_4^4 dr \right] \]
and the asserted inequality follows from \( (L_e) \) and \( (L_\mathscr{T}) \).

4.5 Lemma. For every \( \Delta \in \mathbb{T} \) and every stopping time \( T \leq T_k \) \( (k \geq 0) \)
\[ \mathbb{E} \left[ \sup_{s \in \mathbb{T}} \left\| X^A(s_{T}) - X^A(s_{AT}) \right\|_4^4 \right] \leq D(t) \]
with \( D(t) = 64 t \left( c_1(t)^2 k^4 + c_2(t)^2 \right) + (D_1(t) k^4 + D_2(t)) \).

Proof: For every \( t \geq 0 \) we put \( m(t) := \max \{ k \geq 0 : t_k \leq t \} \). Then
\[ \mathbb{E} \left[ \sup_{s \in \mathbb{T}} \left\| X^A(s_{T}) - X^A(s_{AT}) \right\|_4^4 \right] \]
\[ \leq 8 \mathbb{E} \left[ \sup_{s \in \mathbb{T}} \left\| \int_{s_{AT}}^{s_{T}} g(r, X^A(r)) dr \|_4^4 \right] + 8 \mathbb{E} \left[ \sup_{s \in \mathbb{T}} \left( \int_{s_{AT}}^{s_{T}} \int_{\mathbb{Z}} \left( G(r, z, X^A(r)) \xi \right) (dr, dz) \|_4^4 \right) \right] \]
\[ \leq 8 \sum_{o \leq k < m(t)} (t_{k+1} - t_k)^3 \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left\| g(r, x^A(t_k_{T})) \right\|_4^4 dr \right]
+ 8 \sum_{o \leq k < m(t)} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \int_{\mathbb{Z}} G(r, z, x^A(t_k_{T})) \xi (dr, dz) \|_4^4 \right] \]
\[ \leq 64 \sum_{k} (t_{k+1} - t_k)^3 \left\{ \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left\| g(r, x^A(t_k_{T})) - g(r, x_0) \right\|_4^4 dr \right]
+ \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \int_{\mathbb{Z}} \left( G(r, z, x^A(t_k_{T})) - G(r, z, x_0) \right) \xi (dr, dz) \|_4^4 \right] \right\}
+ 64 \sum_{k} \left\{ \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \int_{\mathbb{Z}} G(r, z, x_0) \xi (dr, dz) \|_4^4 \right] \right\} \]
\[ \leq 64 \sum_{k} (t_{k+1} - t_k)^4 (k^4 c_1(t)^2 + c_2(t)^2)
+ 64 \sum_{k} \left\{ D_1(t) \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left\| X^A(t_k_{T}) - x_0 \right\|_2^2 dr \right] + D_2(t) (t_{k+1} - t_k)^2 \right\}
\leq 64 |\Delta|^2 t (k^4 c_1(t)^2 + c_2(t)^2) + 64 |\Delta| t (D_1(t) k^4 + D_2(t)) \cdot \]

4.6 Lemma. For \( \Delta, \Delta' \in \mathbb{T} \) we put \( T := T_k \wedge T_{k'} \) \( (k \geq 0) \). Then the following inequality holds for \( |\Delta|, |\Delta'| \leq d \):
\[ \mathbb{E} \left[ \sup_{s \in \mathbb{T}} \left\| X^A(s_{T}) - X^A'(s_{AT}) \right\|_4^4 \right] \leq \max(|\Delta|, |\Delta'|) C(t, k) \]
where
\[ C(t, k) = 2^{10} t^3 (t^2 c_1(t)^2 + D_1(t)) \left[ d_2(c_1(t)^2 k^4 + c_2(t)^2) + (k^4 D_1(t) + D_2(t)) \right] \cdot \exp \left[ 3^2 t^2 (t^2 c_1(t)^2 + D_1(t)) \right]. \]

Proof: It follows from lemma 4.5 that

\[ \mathbb{E} \left[ \sup_{s \leq t} \| X^\Delta(s_T) - X^\Delta'(s_T) \| \right] \leq \max(|\Delta|, |\Delta'|) D(t, k) + 27 \mathbb{E} \left[ \sup_{s \leq t} \| X^\Delta(s_T) - X^\Delta'(s_T) \| \right], \]

where the constant \( D(t, k) \) is given by lemma 4.4 and lemma 4.5. Now one can prove - similar as in the proof of theorem 2.2 - that

\[ \mathbb{E} \left[ \sup_{s \leq t} \| X^\Delta(s_T) - X^\Delta'(s_T) \| \right] \leq \mathbb{E} \left[ \sup_{s \leq t} \| X^\Delta(s_T) - X^\Delta'(s_T) \| \right], \]

and the asserted inequality follows again from lemma 2.3.

Proof of theorem 4.1: Let \( \epsilon > 0 \) be given. By lemma 4.3 there exists a \( k > 0 \) such that \( P \left[ T_k < t \right] \leq \epsilon / 4 \) for all \( \Delta \in \mathcal{T} \). Hence \( P \left[ T_k < t \right] \leq \epsilon / 2 \) for \( T_k := T_k(\Delta, \Delta') := T_k \wedge T_k' \). By lemma 4.6 there exists a \( \Delta_o(k) \) such that for all \( \Delta, \Delta' \geq \Delta_o \)

\[ \mathbb{E} \left[ \sup_{s \leq t} \| X^\Delta(s_T_k) - X^\Delta'(s_T_k) \| \right] \leq \epsilon / 2 , \]

and we get

\[ P \left[ \sup_{s \leq t} \| X^\Delta(s) - X^\Delta'(s) \| \geq \epsilon \right] \leq P \left[ T_k < t \right] + P \left[ \sup_{s \leq t} \| X^\Delta(s_T_k) - X^\Delta'(s_T_k) \| \geq \epsilon \right] \leq \epsilon / 2 + \epsilon^{-4} \mathbb{E} \left[ \sup_{s \leq t} \| X^\Delta(s_T_k) - X^\Delta'(s_T_k) \| \right] \leq \epsilon , \]

i.e. the net \( (X^\Delta_t)_{\Delta \in \mathcal{T}} \) converges in probability. It follows by a diagonal argument that there exists a sequence \( (\Delta_n) \) in \( \mathcal{T} \) with \( |\Delta_n| \to 0 \) such that the sequence \( (X^\Delta_n) \) converges P-a.s. on every interval \([0, t]\) uniformly to a process \( X \). This implies that \( X \) has necessarily continuous paths. The decomposition

\[ X - SX = (X - X^\Delta) + (X^\Delta - SX^\Delta) + (SX^\Delta - SX) \]

and an application of lemma 4.5 and an inequality used in lemma 4.6 show that \( X \) is a solution of (1).
References


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Abstract

The existence and uniqueness problems for the pathwise Duncan-Mortensen-Zakai equation of nonlinear filtering are studied following the Robust Equation approach. It is shown here that several examples that include the Kalman-Bucy filtering case can be treated by means of a theorem that solves the Cauchy problem for the Robust Equation.

§1. Introduction

Let \( (\Omega, \mathcal{A}, \mathcal{F}) \) be a probability space and let \{\( \mathcal{A}_t \), \( t \in [0,T] \)\} be an increasing family of \( \sigma \)-algebras of \( \mathcal{A} \). Consider the nonlinear filtering problem with an \( N \)-dimensional state process \( x_t \), and a \( D \)-dimensional observation process \( y_t \):

\[
\begin{align*}
dx_t &= b(t,x_t)dt + \sigma(t,x_t)dw_t, \\
dy_t &= h(t,x_t)dt + dv_t,
\end{align*}
\]

where \( b : [0,T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \), \( \sigma : [0,T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N \), and \( h : [0,T] \times \mathbb{R}^N \rightarrow \mathbb{R}^D \). Here \( (w_t, v_t) = (w_t^1, \ldots, w_t^N, v_t, v_t^1, \ldots, v_t^D) \) is a standard Brownian motion process adapted to \( \{\mathcal{A}_t\} \). The initial state \( x_0 \) is supposed to be an \( N \)-dimensional random vector, measurable with respect to \( \mathcal{A}_0 \), and independent of the Brownian motion \( v_t \).

Let \( \mathcal{Y}_t \) denote the \( \sigma \)-algebra generated by the observations \( y_s \), \( 0 \leq s \leq t \). The goal of Nonlinear Filtering Theory is to study the conditional expectations

\[
\pi_t(f) = \mathbb{E}(f(x_t) | \mathcal{Y}_t)
\]

for suitable real valued functions \( f \). This is because \( \pi_t(f) \) is the best estimate, in quadratic mean sense, of \( f(x_t) \) given the observations \( y_s \), \( 0 \leq s \leq t \).
This estimate depends, in general, nonlinearly on the observations, and it is called the nonlinear filter. When the equations (1.1) and (1.2) are linear, the filter is called the Kalman-Bucy filter, and its study has resulted in a great variety of important applications in engineering (S[1] contains an impressive bibliography).

§2. The Robust Equation approach

Assume that \( x_t \) is a homogeneous Markov process with infinitesimal generator

\[
L = \frac{1}{2} \sum_{i,j=1}^{N} \sum_{k=1}^{M} \sigma_{ik}(t,x)\sigma_{jk}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(t,x) \frac{\partial}{\partial x_i},
\]

(2.1)

where \( \sigma_{ik} \) denotes the \((i,k)\)-th entry of the matrix \( \sigma \), and \( b_i \) denotes the \(i\)-th component of \( b \). M. Zakai (Z[1]) devised the following method to study \( \pi_t \). Let \( \varrho_0 \) be a new probability measure on \((\Omega, \mathcal{A})\) that is equivalent to \( \varrho \), and defined through the Radon-Nikodym derivative

\[
\frac{d\varrho_0}{d\varrho} = \Lambda_t^{-1},
\]

(2.2)

where

\[
\Lambda_t = \exp \left[ \int_0^t \langle h(s,x_s), dy_s \rangle - \frac{1}{2} \int_0^t \langle h(s,x_s), h(s,x_s) \rangle ds \right],
\]

(2.3)

and \( \langle , \rangle \) denotes inner product in \( \mathbb{R}^n \).

By a theorem due to Girsanov (W[1], p. 232) the new space \((\Omega, \mathcal{A}, \varrho_0)\) is a probability space, and the process \((x_t, y_t)\) has the same distribution under \( \varrho_0 \) as the process \((x_t, v_t)\) has under \( \varrho \).

If \( \int_0^T \langle h(s,x_s), h(s,x_s) \rangle ds \leq \alpha \) a.s. \( \varrho \), and if \( f \) is a bounded Borel function on \( \mathbb{R}^N \), then

\[
\pi_t(f) = \frac{\mathbb{E}_0(f(x_t) \Lambda_t | Y_t)}{\mathbb{E}_0(\Lambda_t | Y_t)},
\]

(2.4)

where \( \mathbb{E}_0 \) is the expectation respect to \( \varrho_0 \).

Hence, the problem of studying \( \pi_t(f) \) is transformed into the problem of studying the unnormalized conditional expectation

\[
\sigma_t(f) = \mathbb{E}_0(f(x_t) \Lambda_t | Y_t)
\]

(2.5)
under $\mathcal{P}_0$. The statistic $\sigma_t(f)$ is essentially the same as $\pi_t(f)$. The only difference between them is a constant of normalization. The advantage of this approach is that $\sigma_t(f)$ satisfies a linear stochastic differential equation while the stochastic differential equation satisfied by $\pi_t(f)$ is nonlinear. (Cf. DM[1]). Zakai proved rigorously under a number of hypotheses that if $f : \mathbb{R}^N \to \mathbb{R}$ is a $C^2$ function, then $\sigma_t(f)$ satisfies

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Lf) \, ds + \int_0^t \sigma_s(hf) \, dy_s,$$

(2.6)

where $\sigma_s(hf) = (\sigma_s(h_1 f), \ldots, \sigma_s(h_d f))$.

Next, suppose that the conditional distribution of $x_t$ given $\mathcal{F}_t$ has a density $\tilde{p}(t,x)$, and define the unnormalized conditional density as

$$p(t,x) = \sigma_t(1) \tilde{p}(t,x).$$

(2.7)

Zakai proved also that, under certain hypotheses, $p(t,x)$ satisfies the stochastic partial differential equation in Ito form

$$dp(t,x) = L^* p(t,x) \, dt + p(t,x) \langle h(t,x), dy_t \rangle,$$

(2.8)

where $L^*$ is the formal adjoint of the partial differential operator $L$. This expression is called the Duncan-Mortensen-Zakai equation.

Some ideas of Sussmann (S[2]) and Doss (D[1]), were followed by Davis (D[2]) to give a pathwise interpretation of (2.8). First, rewrite (2.8) in Stratonovich form

$$dp(t,x) = (L^* - \frac{1}{2} \langle h(t,x), h(t,x) \rangle) p(t,x) \, dt + p(t,x) \langle h(t,x), dy_t \rangle,$$

(2.9)

The pathwise version of (2.9) is the expression

$$\frac{dp(t,x)}{dt} = (L^* - \frac{1}{2} \langle h(t,x), h(t,x) \rangle) p(t,x) + p(t,x) \langle h(t,x), \frac{dy_t}{dt} \rangle,$$

(2.10)

where $y(\cdot)$ is a path of the observation process $Y_t$. But this process has the sample path properties of Brownian motion (since the process $Y_t$ is a Brownian motion process with respect to $\mathcal{P}_0$, and $\mathcal{P}_0$ is absolutely continuous with respect to $\mathcal{P}$). Therefore, $\frac{dy_t}{dt}$ is defined only for a family of $y$'s that has zero measure. One wants to give a meaning to (2.10) for all $y$'s in a set of full measure as is any of the sets $C^p[0,T]$, of Hölder continuous functions, with $0 < \alpha < \frac{1}{2}$. For this, consider the transformation
\[ q(t,x) = \exp(-h(t,x),y(t)>)p(t,x). \] \hspace{1cm} (2.11)

Then, if \( \sigma, h \) and \( b \) are sufficiently smooth, the new function \( q \) satisfies the partial differential equation

\[ \frac{\partial q}{\partial t}(t,x) = \exp(-h(t,x),y(t)>)\left( L^* - \frac{1}{2}h(t,x),h(t,x) > - \frac{\partial h}{\partial t}(t,x),y(t) > \right) \]

\[ \left( \exp(h(t,x),y(t)>)q(t,x) \right). \] \hspace{1cm} (2.12)

For the general nonlinear filtering problem (1.1), (1.2), this partial differential equation, called the Robust Equation, is of second order and of degenerate parabolic type with coefficients that are unbounded in the space variables and that depend on the paths \( y(t) \), but do not depend on \( \frac{dy}{dt} \).

A concept of solution of (2.10) can be defined by saying that \( p(t,x) \) solves (2.10) if \( q(t,x) \) defined by (2.11) is a solution of (2.12) with initial condition \( q(0,x) = \psi(x) \), the density of the random vector \( x_0 \). This is the basis for the results obtained by Baras et al. (BBM[1], BBH[1], H[1]). Their theorems are based on results on parabolic P.D.E.'s due to Aronson and Besala (AB[1], B[1], and B[2]). So, in order to apply these results, it is necessary to suppose that \( L^* \) is elliptic, or in other words, that the matrix \( A(t,x) \) with

\[ a_{ij}(t,x) = \sum_{k=1}^{M} a_{ik}(t,x) \sigma_{jk}(t,x) \] \hspace{1cm} (2.13)

is positive definite. These theorems have the virtue of allowing the coefficients of (2.12) to have somewhat relaxed conditions of growth. On the other hand, the hypothesis of an elliptic \( L^* \) is not natural in the theory of filtering. First, even in the well known case of linear filtering where \( \sigma \) is just a function of \( t \), it is unnatural to suppose that \( A(t) \) is positive definite, or even nonzero, for all times. Second, it may be the case that \( M < N \). If so, \( A \) cannot be positive definite.

Radically different approaches to study (2.8) have been followed in P[1], P[2], and KK[1]. Pardoux studied (2.8) as a stochastic partial differential equation. Kallianpur and Kallendar considered a different model than (1.1), (1.2). In fact, they studied the case where instead of a Brownian motion \( v_t \) in (1.2) one has additive white noise. In both cases it is necessary to make the hypothesis of a positive definite matrix \( A \). Also, various kinds of restrictive conditions on growth in \( x \) for the coefficients \( \sigma, b \) and \( h \) are made.
Also, the pathwise equation (2.10) has been studied by means of probabilistic methods in \( S[4] \).

§3. Results on P.D.E.'s

We base our work on a theorem that does not require \( A(t,x) \) to be positive definite, and that only requires \( y(\cdot) \) to be continuous. To state it we introduce the following notation and definitions.

The symbol \( Z_+ \) will denote the set of nonnegative integers, i.e.,
\[ Z_+ = \mathbb{N} \cup \{0\} \]. Let \( Z_+^N \) denote the cartesian product of \( N \) copies of \( Z_+ \). If \( \alpha = (\alpha_1, \ldots, \alpha_N) \in Z_+^N \), then we write \( |\alpha| = \sum_{i=1}^{N} \alpha_i \), and we let \( \partial_{\alpha}^{x} \) be the partial differential operator of order \( |\alpha| \) given by
\[
\partial_{\alpha}^{x} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \ldots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}.
\]

Definition 1:

A function \( \phi : \mathbb{R}^N \to \mathbb{R} \) is said to belong to \( C_0^\infty (\mathbb{R}^N) \) if it is of class \( C^\infty \) on \( \mathbb{R}^N \), and \( \forall \alpha \in Z_+^N \) the partial derivative \( \partial_{\alpha}^{x} \phi \) is bounded.

Definition 2:

A function \( f : [a,b] \times \mathbb{R}^N \to \mathbb{R} \) is said to belong to the space \( D_b([a,b] \times \mathbb{R}^N) \) if
a) \( f \) is of class \( C^\infty \) as a function of \( x \),
b) \( \forall \alpha \in Z_+^N \), \( \partial_{\alpha}^{x} f \) is bounded on \([a,b] \times \mathbb{R}^N\), and
c) \( \forall \alpha \in Z_+^N \), \( \partial_{\alpha}^{x} f \) is of class \( C^1 \) on \([a,b] \times \mathbb{R}^N\).

Consider the second order partial differential equation of degenerate parabolic type
\[
\frac{\partial q}{\partial t}(t,x) = \frac{1}{2} \sum_{k=1}^{M} X_k^2(t,x)q(t,x) + Y(t,x)q(t,x) + c(t,x)q(t,x), \quad (3.1)
\]
where \( (t,x) \in [0,T] \times \mathbb{R}^N \), \( Y \) is the time dependent vector field.
\[ Y(t,x) = \sum_{i=1}^{N} \eta_i(t,x) \frac{\partial}{\partial x_i}, \]  
\[ \text{and, for } k = 1,2,\ldots,M, X_k \text{ is the time dependent vector field} \]
\[ X_k(t,x) = \sum_{i=1}^{N} \sigma_{ik}(t,x) \frac{\partial}{\partial x_i}, \]  
We note here that
\[ X_k^2 = \sum_{i,j=1}^{N} \sigma_{ik} \sigma_{jk} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \mu_{ik} \frac{\partial}{\partial x_i}, \]  
where \( \sigma = (\sigma_{ik}) \) is the diffusion coefficient matrix of \( x_t \), and
\[ \mu_{ik} = \sum_{j=1}^{N} \sigma_{jk} \frac{\partial}{\partial x_j}. \]  
Let \( a_{ij} = \sum_{k=1}^{M} \sigma_{ik} \sigma_{jk} \) be the local covariance matrix of \( x_t \).

Consider the following conditions on the coefficients of (3.1).

(H1) The functions \( \sigma_{ik}, \eta_i \) and \( c \) are defined on \([0,T] \times \mathbb{R}^N\). They are of class \( C^\infty \) in \( x \), and jointly continuous in \( t \) and \( x \), together with all the partial derivatives with respect to \( x \) of all orders.

(H2) \( \forall \alpha \in \mathbb{Z}_+^N \text{ with } |\alpha| > 1, \text{ the functions } \partial^\alpha x_{ik}, \text{ and } \partial^\alpha x_{ik} \text{ are bounded on } [0,T] \times \mathbb{R}^N. \)

(H3) \( \forall \alpha \in \mathbb{Z}_+^N \text{ with } |\alpha| > 2, \text{ the functions } \partial^\alpha x_{ij} \text{ are bounded on } [0,T] \times \mathbb{R}^N. \)

(H4) \( \forall \alpha \in \mathbb{Z}_+^N \text{ with } |\alpha| > 1, \forall k > 0, \forall i < N, \text{ the expression} \]
\[ k(|\partial^\alpha x_{ik}(t,x)| + |\partial^\alpha x_{ik}(t,x)| + \frac{|\eta_i(t,x)|}{1 + |x|} + c(t,x) \]
\[ \text{is bounded above on } [0,T] \times \mathbb{R}^N. \]

(H5) Let \( t + H(\tau,t,x) \) be the integral curve of the vector field \( -Y \) that passes through \((\tau,x)\). If there is a time \( t_0 \in [0,T], \) and \( x \in \mathbb{R}^N \) such that \( H(t_0,t,x) \) is defined for \( t \in [t_0,t_1] \) but not for \( t_1 \) (i.e., when \( \lim_{t \to t_1} |H(t_0,t,x)| = \infty \)), then
\[ \lim_{t \to t_1} \int_{t_0}^{t} c(\tau,H(\tau,t,x))d\tau = -\infty. \]

We are now ready to state the first theorem.
Theorem 1:

Suppose that (H1) through (H5) hold. Then, given $\phi \in C^\infty_b(\mathbb{R}^N)$, equation (3.1) has a unique solution $r : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that

$$r \in C^\infty_b([0,T] \times \mathbb{R}^N), \quad \text{and} \quad r(0,x) = \phi(x).$$

Under certain additional hypotheses on the coefficients, the solution of the partial differential equation (3.1) depends continuously on the observation path $t + y(t)$, and on the initial condition $\phi$. Rigorously, we suppose that

(Hy) $\{y_m \}_{m=1}^\infty$ and $y$ are continuous functions defined on $[0,T]$, with values in $\mathbb{R}^N$, and such that $\lim_{m \to \infty} \sup_{t \in [0,T]} |y_m(t) - y(t)| = 0$, as $m \to \infty$.

Let $r$ denote the solution in $D_b([0,T] \times \mathbb{R}^N)$ of equation (3.1) with initial condition $\phi(x)$. Let $r^m$ denote the solution in $D_b([0,T] \times \mathbb{R}^N)$ of equation (3.1) where $y$ is replaced by $y^m$, and with initial condition $\phi^m(x)$. Then, the function

$$v^m = r^m - r$$

belongs to $D_b([0,T] \times \mathbb{R}^N)$ and satisfies

$$\frac{\partial v^m}{\partial t} = \frac{1}{2} \sum_{k=1}^M X_k v^m + \nabla v^m + \nabla^2 v^m + q^m r^m,$$  

where

$$q^m(t,x) = \sum_{i=1}^N \left( \eta_i^m(t,x) - \eta_i(t,x) \right) \frac{\partial^2}{\partial x_i^2} + c^m(t,x) - c(t,x),$$

$$\eta_i^m - \eta_i = \sum_{j=1}^N \sum_{i,j} a_{ij} (y^m - y, \frac{\partial h}{\partial x_j}) + \frac{\partial h}{\partial x_j}, \quad \text{and}$$

$$c^m - c = \frac{1}{2} \sum_{i,j=1}^N a_{ij} \left[ \langle y^m - y, \frac{\partial h}{\partial x_j} \rangle - \langle y^m - y, \frac{\partial h}{\partial x_i} \rangle - \langle y^m - y, \frac{\partial h}{\partial x_j} \rangle - \langle y^m - y, \frac{\partial h}{\partial x_i} \rangle \right]$$

$$+ \sum_{i=1}^N \sum_{j=1}^N \sum_{j=1}^N \left( \frac{3a_{ij}}{2} \right) \langle y^m - y, \frac{\partial h}{\partial x_j} \rangle - \langle y^m - y, \frac{\partial h}{\partial x_i} \rangle - \langle y^m - y, \frac{\partial h}{\partial x_j} \rangle - \langle y^m - y, \frac{\partial h}{\partial x_i} \rangle.$$

The function $v^m$ satisfies the initial condition

$$v^m(0,x) = \phi^m(x) - \phi(x).$$

Besides hypotheses (H1) through (H5), we suppose the following:
The functions \( h, b \) and \( \frac{\partial h}{\partial t} \) of the nonlinear filtering problem (1.1), (1.2) are of polynomial growth in \( x \), uniformly for \( t \in [0,T] \), together with all their partial derivatives with respect to \( x \) of all orders.

Let \( f^m(t,x) \) denote any of the coefficients of \( Q^m \). Then \( \forall k > 0, \forall a \in \mathbb{Z}^N_+ \), \( k|a|f^m(t,x) + c(t,x) \) is bounded above on \([0,T] \times \mathbb{R}^N\).

We can state now the second theorem.

**Theorem 2:**

Suppose that (H1) through (H7) hold. Then, if (Hy) and (Hφ) hold, the solutions \( r^m \) converge to the solution \( r \), uniformly on \([0,T] \times \mathbb{R}^N\), together with all the \( x \)-derivatives of all orders.

A detailed proof of theorems 1 and 2 is lengthy. It is contained in the doctoral thesis F[1], and will appear elsewhere in F[2]. We shall show here that their hypotheses allow for applications to rich classes of filtering problems.

§4. Examples.

This is the main section of our paper. We show here that theorems 1 and 2 apply to the Robust Equation (2.12) that corresponds to several important examples of filtering problems.

Throughout this section we suppose that the path \( y(*) \) that appears in (2.12) is continuous on \([0,T]\).


Consider the nonlinear filtering problem defined on \([0,T]\) by the model (1.1) and (1.2). We suppose that

- (1.i) \( M, N \) and \( D \) are arbitrary,
- (1.ii) \( h \) is \( C^1 \) as a function of \( t \),
- (1.iii) \( b, c, h \) and \( \frac{\partial h}{\partial t} \) are \( C^\infty \) functions of the space variable \( x \), with continuous partial derivatives with respect to \( x \) of all orders, and
- (1.iv) \( \forall o \in \mathbb{Z}^N_+ \), the derivatives \( \frac{\partial^o b}{\partial x^o}, \frac{\partial^o c}{\partial x^o}, \frac{\partial^o h}{\partial x^o}, \text{ and } \frac{\partial^o}{\partial x^o} \frac{\partial h}{\partial t} \) are bounded on \([0,T] \times \mathbb{R}^N\).

Then the Robust Equation (2.12) satisfies hypotheses (H1) through (H5). In fact, (H1) through (H4) are trivial consequences of (1.i)-(1.iv) and of the fact that the observation path \( y(*) \) is continuous. The hypothesis (H5) holds trivially because \( Y \) has bounded coefficients, therefore its integral curves never blow up.
Also, hypotheses (H6) and (H7) hold. Then, theorems 1 and 2 apply.

(2) Linear Filtering.

Consider the linear filtering problem defined on \([0, T]\) by (1.1) and (1.2) with \(M, D,\) and \(N\) arbitrary, and

\[
\begin{align*}
(2.i) \quad & b(t, x) = F(t)x, \\
(2.ii) \quad & \sigma(t, x) = G(t), \\
(2.iii) \quad & h(t, x) = H(t)x,
\end{align*}
\]

where \(F(t) = (F_{ij}(t))\) is an \(N \times N\) matrix with continuous entries, \(G(t) = (G_{ik}(t))\) is an \(N \times M\) matrix with continuous entries, and \(H(t) = (H_{ij}(t))\) is a \(D \times N\) matrix with \(C^1\) entries.

This is the second example that shows that degeneracies of the matrix \(A(t, x) = (a_{ij}(t, x))\) (in this case \(A\) is the product of \(G\) with its transpose, and it does not depend on \(x\)) appear very naturally in filtering theory.

In this case the equation (2.12) does not necessarily satisfy the hypotheses of theorems 1 and 2. But if we introduce a new transformation

\[
\tilde{q}(t, x) = \exp(B(t, x))q(t, x),
\]

then \(\tilde{q}\) satisfies

\[
\frac{\partial \tilde{q}}{\partial t} = \frac{1}{2} \sum_{k=1}^{M} X_k^2 \tilde{q} + Y \tilde{q} + c \tilde{q},
\]

where

\[
\begin{align*}
X_k^2 &= \sum_{i,j=1}^{N} C_{ik}(t)G_{jk}(t)\frac{\partial^2}{\partial x_i \partial x_j}, \\
Y &= \sum_{i=1}^{N} \eta_i(t, x)\frac{\partial}{\partial x_i},
\end{align*}
\]

\[
\eta_i = \sum_{j=1}^{D} a_{ij}(t)[\sum_{\ell=1}^{N} \frac{\partial}{\partial x_{\ell}} y_{\ell}(t)H_{ij}(t) - \frac{\partial B}{\partial x_j} - \sum_{j=1}^{D} F_{ij}(t)x_j],
\]

and

\[
c = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(t)[\sum_{\ell=1}^{D} \frac{\partial}{\partial x_{\ell}} y_{\ell}(t)H_{ij}(t) - \frac{\partial B}{\partial x_j} - \sum_{j=1}^{D} F_{ij}(t)x_j]
\]

\[
- \sum_{i=1}^{N} j=1 \sum_{j=1}^{D} \frac{\partial}{\partial x_i} (F_{ij}(t)x_j)[\sum_{\ell=1}^{N} \frac{\partial}{\partial x_{\ell}} y_{\ell}(t)H_{ij}(t) - \frac{\partial B}{\partial x_j} - \sum_{j=1}^{D} F_{ij}(t)],
\]
The question now in how to choose $B$ so that hypotheses (H1) through (H7) hold. There are two cases to be considered. If the matrix $H(t)$ is non-degenerate, then $B = 0$ is the choice. This is because $n$ will be of linear growth in $|x|$ and $c$ will be quadratic in $|x|$ with a negative coefficient multiplying $|x|^2$. The second case occurs when no hypothesis about the degeneracies of $H(t)$ are made. Thus, the term $-\frac{1}{2} \sum_{g=1}^{D} \sum_{j=1}^{N} H_{gj}(t)x_j^2$ can not be used to dominate the linear terms of $c$. In this second case, we define

$$B(t,x) = |x|^2 e^{-\alpha t},$$

where $\alpha$ is a positive constant that will be chosen later. Then

$$\frac{\partial B}{\partial t} = -\alpha |x|^2 e^{-\alpha t}, \quad \frac{\partial B}{\partial x} = 2x e^{-\alpha t} \quad \text{and} \quad \frac{\partial^2 B}{\partial x^2} = 2e^{-\alpha t}.$$  

Thus $c$ is quadratic in $|x|$, $\partial_x^\alpha c$ for $|\alpha| = 1$ and $n$ are of linear growth in $|x|$, and $\partial_x^\beta c$ and $\partial_x^\gamma n$ are bounded in norm for every $|\beta| > 2, |\gamma| > 1$. The terms of $c$ that are quadratic in $|x|$ are

$$Q = \frac{1}{2} \sum_{i,j=1}^{D} a_{ij}(t) \frac{\partial B}{\partial x_i} \frac{\partial B}{\partial x_j} + \sum_{i,j=1}^{N} F_{ij}(t)x_j \frac{\partial B}{\partial x_i}$$

$$- \frac{1}{2} \sum_{g=1}^{D} \sum_{j=1}^{N} H_{gj}(t)x_j^2 + \frac{\partial B}{\partial t}.$$

Since $A$ and $F$ are continuous on $[0,T]$, then there is a constant $k > 0$ such that

$$Q < k |x|^2 e^{-2\alpha t} + k|x|^2 e^{-\alpha t} - \alpha |x|^2 e^{-\alpha t} =$$

$$= e^{-\alpha t}|x|^2(k e^{-2\alpha t} + k - \alpha).$$

Since $e^{-\alpha t} < 1$ for every $\alpha > 0$, we can select $\alpha > 0$ so that $(k e^{-\alpha t} + k - \alpha) < 0$. Then, $c$ turns out to be bounded above, and all the hypotheses of Theorems 1 and 2 hold.

Again in this case, the hypothesis (H.5) is trivially satisfied since the integral curves of the vector field $Y$ never blow up. (Because $n$ is of linear growth in $|x|$.)
A nonlinear filtering problem with unbounded coefficients and degeneracy in the diffusion.

Consider the nonlinear filtering problem defined for \( t \in [0, T] \) by

\[
\begin{align*}
(3.\, i) & \qquad dX_1 = dW_1, \\
(3.\, ii) & \qquad dX_2 = X_1 \, dW_2, \\
(3.\, iii) & \qquad dY = X_2 \, dt + dv.
\end{align*}
\]

Then, referring to the model (1.1)-(1.2), we are setting \( N = M = 2 \), and \( D = 1 \). The results that theorems 1 and 2 give in this case were announced in FS[1].

This is another example with a degenerate \( A \), for in this case

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & x_1^2 \end{bmatrix}
\]

has vanishing determinant for \( x_1 = 0 \). Also, in this example, equation (2.12) does not satisfy all of (H1)-(H7). Again a transformation has to be introduced.

Let

\[
\tilde{q}(t,x) = \exp(B(t,x))q(t,x).
\]

Then

\[
\frac{\partial \tilde{q}}{\partial t} = \frac{1}{2} \frac{\partial^2 \tilde{q}}{\partial x_1^2} + \frac{1}{2} x_1^2 \frac{\partial^2 \tilde{q}}{\partial x_2^2} - \frac{\partial B}{\partial x_1} \frac{\partial \tilde{q}}{\partial x_1} + (x_1^2 y - x_1^2 \frac{\partial B}{\partial x_2}) \frac{\partial \tilde{q}}{\partial x_2} + c \tilde{q},
\]

where

\[
c = \frac{1}{2} \left( \frac{\partial B}{\partial x_1} \right)^2 - \frac{\partial B}{\partial x_1} \frac{\partial B}{\partial x_2} + \frac{1}{2} x_1^2 \frac{\partial B}{\partial x_2} + \frac{1}{2} x_2^2 + \frac{\partial B}{\partial t}.
\]

If \( B \) were zero, then \( c \) would not be bounded above because the term \( \frac{1}{2} x_1^2 y^2 \) could not be controlled. So we are forced to look for a nontrivial \( B \) in order to fulfill the hypothesis (H4).

Let \( B : [0, T] \times \mathbb{R}^2 \times \mathbb{R} \) be given by

\[
B(t,x_1,x_2) = -t a x_1^2,
\]

where \( a \) is a positive constant that will be chosen below. Then

\[
- \frac{\partial B}{\partial x_1} = 2 t a x_1^1,
\]
\[ x_1^2 y - x_1^2 \frac{\partial B}{\partial x_2} = x_1^2 y, \]

and

\[ c = 2t^2 \alpha x_1^2 + t\alpha + \frac{1}{2} x_1^2 y^2 - \frac{1}{2} x_2^2 - \alpha x_1^2. \]

The only hypothesis that requires some work is (H4). In fact, we need to show that \( c \) is bounded above. For this, let us look at the coefficient \( Q \) that multiplies \( x_1^2 \). That is

\[ Q = 2t^2 \alpha^2 + \frac{\lambda^2}{2} - \alpha. \]

Since \( y \) is continuous on \([0,T]\), it is bounded. Let \( \lambda = \max_{t \in [0,T]} \frac{y^2(t)}{2} \), and let \( \alpha > 0 \) be chosen so that \( \lambda - \alpha < 0 \). Then if we take \( T_1 \) such that

\[ 2\alpha^2 T_1^2 < \alpha - \lambda, \]

we obtain \( Q < 0 \). Given \( \alpha, T_1 \) has to satisfy

\[ T_1 < \frac{1}{2\alpha} - \frac{\lambda}{2\alpha^2} = T_0. \]

The maximum \( T_0 \) as a function of \( \alpha \) is attained when \( \frac{1}{2} = \frac{\lambda}{\alpha} \), i.e., when \( \alpha = 2\lambda \). This gives

\[ T_0 = \frac{1}{4\lambda}. \]

Then, since \( Q < 0 \) for \( t \in [0,T_0] \) it is easy to check that all the hypotheses of Theorems 1 and 2 hold for \( t \in [0,T_0] \).

(4) The cubic sensor.

Consider the nonlinear filtering problem defined for \( t \in [0,T] \) by

\[
\begin{align*}
\frac{dx}{dt} &= dw, \\
\frac{dy}{dt} &= x^3 dt + dv.
\end{align*}
\]

Here \( D = M = N = 1 \). Then the Robust Equation (2.12) is

\[
\frac{\partial q}{\partial t} = \frac{1}{2} \frac{\partial^2 q}{\partial x^2} + y(t) 3x^2 \frac{\partial q}{\partial x} + \frac{1}{2} [y^2(t) 9x^4 + y(t) 6x - x^6] q.
\]

Then (H1) through (H4) are easy to check. Basically they follow because \( c \) contains the term \( -\frac{x^6}{2} \). Also (H6) is trivial, and (H7) follows because the coefficients of \( Q^m \) are of degree \( < 4 \) in \( x \).

Finally, it requires some work in check (H5). In fact, since \( \eta = 3yx^2 \),
then it is conceivable that the integral curves of $Y = \frac{\partial}{\partial x}$ might blow up. It is easy to check that the flow $F$ of the vector field $Y$ is given by

$$F(t,s,x) = \begin{cases} 
-\frac{1}{3\int_0^t y(\xi)d\xi - \frac{1}{x}} & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases}$$

Suppose that there is a $t_0 \in [0, T]$, and an $x \in \mathbb{R}^N$ such that

$$\lim_{t \to t_0} \left| \int_0^t \frac{1}{3\int_0^t y(\xi)d\xi - \frac{1}{x}} \right| = \infty. \quad (4.1)$$

We want to show that

$$\lim_{t \to t_0} \int_0^t c(\tau, F(\tau, t, x)) d\tau = -\infty. \quad (4.2)$$

Since $c(t, x) = \frac{1}{2} [y^2(t)9x^4 + y(t)6x - x^6]$, in order to prove (4.2) it is enough to show that

$$\lim_{t \to t_1} \int_0^t \left| \frac{1}{3\int_0^t y(\xi)d\xi - \frac{1}{x}} \right| d\tau = \infty.$$ 

But

$$3\int_0^t y(\xi)d\xi - \frac{1}{x} = \alpha(t) + \beta(t),$$

where

$$\alpha(t) = 3\int_0^t y(\xi)d\xi - \frac{1}{x},$$

and

$$\beta(t) = 3\int_0^t y(\xi)d\xi.$$ 

Let

$$\tilde{\alpha}(t) = \sup\{|\alpha(s)| : t < s < t_1\}.$$ 

Then $\tilde{\alpha}(t)$ is decreasing, and in view of (4.1), it converges to zero as $t \to t_1^-$. Then, since $y$ is continuous, $\exists \kappa > 0$ such that

$$|\alpha(t) + \beta(t)| < |\alpha(t)| + \kappa(t - t_0) < \tilde{\alpha}(t) + \kappa(t - t_0).$$
Then, the Monotone Convergence Theorem implies that

\[ \lim_{t \to t_1} \int_{t_0}^{t} \frac{1}{3 \int_{t_0}^{t} \gamma(\xi) d\xi - \frac{1}{x}} \frac{d\tau}{6} = \int_{t_0}^{t_1} \left( \frac{1}{x(\tau - t_0)} \right)^6 d\tau = \infty, \]

as was to be shown.

Thus, all the hypotheses of Theorems 1 and 2 hold for the cubic sensor.

We have shown several filtering problems to which Theorems 1 and 2 apply. To finish, we show an example for which theorems 1 and 2 do not apply. At first glance, this example seems to present the same difficulties of the cubic sensor, but it is not so.

(5) The two dimensional cubic sensor.

Consider the nonlinear filtering problem defined by

\begin{align*}
(5i) \quad & dx_1 = dw_1, \\
(5ii) \quad & dx_2 = dw_2, \\
(5iii) \quad & dy = (x_1^3 + x_2^3) dt + dv.
\end{align*}

Here \( D = 1, M = N = 2 \). Then the Robust Equation (2.11) is

\[ \begin{aligned}
\frac{\partial q}{\partial t} &= \frac{1}{2} \left( \frac{\partial^2 q}{\partial x_1^2} + \frac{\partial^2 q}{\partial x_2^2} \right) + 3y x_1^2 \frac{\partial q}{\partial x_1} + 3y x_2^2 \frac{\partial q}{\partial x_2} + \\
&\quad + 3y x_1 q + 3y x_2 q + \frac{9}{2} y^2 x_1^4 q + \frac{9}{2} y^2 x_2^4 q - \frac{1}{2} (x_1^3 + x_2^3)^2 q.
\end{aligned} \]

The hypothesis \((H_4)\) doesn't hold in this case. (Basically because \(- \frac{1}{2} (x_1^3 + x_2^3)^2\) vanishes when \( x_1 = -x_2 \), thus \( c \) is unbounded above). Theorems 1 and 2 do not apply to this example. As a matter of fact, deep negative results have been proved for the two dimensional cubic sensor (S[3]).

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SAMPLE CONTINUITY MODULI THEOREM IN VON NEUMANN ALGEBRAS

by

S. Goldstein and A. Tuczek

Let $M$ be a von Neumann algebra with a normal faithful semifinite trace $\tau$, and $\tilde{M}$ - the algebra of operators measurable in Nelson's sense. We recall that $\tilde{M}$ consists of closed densely defined operators $x$ affiliated with $M$ such that $\tau(e_{\lambda,\infty}(|x|)) < \infty$ for some $\lambda > 0$, where $e_{\lambda}(z)$ is the spectral projection of $z$ corresponding to the Borel subset $E$ of the line.

The basic role in our consideration is played by two notions of convergence: convergence in measure and convergence in Segal's sense. The measure topology in $\tilde{M}$ is given by the fundamental system of neighborhoods of zero of the form

$$N(\varepsilon, \delta) = \{x \in \tilde{M} : \text{there exists a projection } p \text{ in } M \text{ such that } xp \in M, \|xp\| \leq \varepsilon \text{ and } \tau(p^+) \leq \delta\}.$$  

It follows that $\tilde{M}$ endowed with the measure topology is a topological $\ast$-algebra (see [4] for a more detailed description of $\tilde{M}$). The following form of convergence in measure, similar to the classical one, will suit best for our purposes (see [6], Prop.2.7): 

$$x_n \to x \text{ in measure if and only if, for each } \varepsilon > 0, \quad \tau(e_{[\varepsilon,\infty]} |x_n - x|) \to 0.$$  

Following Lance [3], we shall say that a sequence $\{x_n\}$ of elements from $\tilde{M}$ converges to $x \in \tilde{M}$ in Segal's sense if, for each $\varepsilon > 0$, there is a projection $p$ in $M$ with $\tau(p^+) < \varepsilon$, such that 

$$(x_n - x)p \in M \text{ for sufficiently large } n, \text{ and } \|x_n - x\|p \to 0 \text{ (cf. with weaker forms of convergence in [1],[5]).}$$  

Now, let $X$ be a mapping $X : [a,b] \to \tilde{M}$. The "continuity" of $X$ in Segal's sense is defined in the obvious way. It is known that if $X$ is "continuous" in Segal's sense, then it is continuous in measure (since Segal's convergence implies convergence in measure). We shall seek for some conditions giving the reverse implication. It turns out that there exist conditions on the rate of convergence in measure which
give even more, namely, the uniform "continuity" in Segal's sense. To make our considerations clearer, let us refer to the commutative case and consider a stochastic process \( \xi : [a,b] \to L^0(\Omega,F,P) \) where \( (\Omega,F,P) \) is a probability space and \( L^0(\Omega,F,P) \) stands for the set of random variables on \( \Omega \). In this case we can speak of the continuity of \( \xi \) with probability one, which means that \( \xi(t) \to \xi(t_0) \) as \( t \to t_0 \) \( P \)-almost everywhere. According to the Egorov theorem, this is equivalent to the condition: for each \( \epsilon > 0 \), there is a set \( E \in F \) with \( P(E) > 1 - \epsilon \) such that \( \sup_{\omega \in E} |\xi(t_n,\omega) - \xi(t_0,\omega)| \to 0 \) as \( t_n \to t_0 \). Now, the condition of "uniform continuity with probability one" becomes clear and takes the form:

for each \( \epsilon > 0 \), there is a set \( E \in F \) with \( P(E) > 1 - \epsilon \) such that, for each \( \eta > 0 \), there is \( \delta > 0 \) satisfying: if \( s,t \in [a,b] \) and \( |s - t| < \delta \), then \( \sup_{\omega \in E} |\xi(s,\omega) - \xi(t,\omega)| < \eta \).

Let us note that this condition implies, in particular, the continuity of trajectories of \( \xi \) with probability one.

Returning to the non-commutative case, we shall say that \( X \) is uniformly continuous in Segal's sense if,

for each \( \epsilon > 0 \), there is a projection \( p \) in \( M \) with \( \tau(p) < \epsilon \) such that, for each \( \eta > 0 \), there is \( \delta > 0 \) satisfying: if \( s,t \in [a,b] \) and \( |s - t| < \delta \), then \( \| [X(s) - X(t)]p \| < \eta \).

The importance of the notion of separability for a stochastic process is widely known. We shall show that in the non-commutative case, while considering uniform Segal continuity, the "process" is, under mild assumptions, always "separable".

**PROPOSITION 1.** Let \( X \) be a mapping \( X : [a,b] \to \tilde{M} \) and \( T \) - an arbitrary dense subset of \( [a,b] \). Assume that \( X \) is left (right) continuous in measure. If (1) is satisfied for \( s,t \in T \), then \( X \) is uniformly continuous in Segal's sense.

**Proof.** Take an arbitrary \( \epsilon > 0 \) and choose, according to (1), a sequence \( p_n \) of projections in \( M \) with \( \tau(p_n^2) < \epsilon/2^n \) and positive numbers \( \delta_n \) such that, for \( t', t'' \in T \), \( |t' - t''| < \delta_n \) implies \( \| [X(t') - X(t'')]p_n \| < \epsilon/2^n \). Put \( p = \bigwedge p_n \). Then
\[ \tau(p^t) \leq \sum_{n=1}^{\infty} \tau(p^t) < \epsilon. \]

Let us now assume that \( X \) is left continuous in measure and let \( \{t_n\} \) be a sequence of elements from \( T \) such that \( t_n \to t^- \). Consider the sequence \( \{X(t_n)p]\). Given \( \gamma > 0 \), choose \( N \) such that \( \frac{\epsilon}{2^N} < \gamma \). For this \( N \), take \( K \) such that \( |t_k - t_l| < \delta \) for \( k, l \geq K \). We then have

\[ X(t_k)p - X(t_l)p = [X(t_k) - X(t_l)]pNp \in M \quad \text{and} \quad \|X(t_k)p - X(t_l)p\| \leq \|X(t_k) - X(t_l)\|pNp < \frac{\epsilon}{2^N} < \gamma. \]

Thus the sequence \( \{X(t_n)p\} \) is Cauchy in norm. From the assumed continuity of \( X \) and the continuity of algebraic operations in \( \tilde{M} \) (see [4, Th.1]) it follows that \( X(t_n)p \to X(t)p \) in measure, thus \( X(t_n)p \to X(t)p \) in norm.

Let now \( n > 0 \) be arbitrary. Take \( \delta > 0 \) as in (1). For any \( s, t \in (a,b) \) satisfying \( |s - t| < \delta \), choose sequences \( \{s_n\} \subset T, \{t_n\} \subset T \) with \( s_n \to s^- \), \( t_n \to t^- \). We have \( |s_n - t_n| < \delta \) for sufficiently large \( n \), \( X(s_n)p \to X(s)p \) in norm and \( X(t_n)p \to X(t)p \) in norm. Hence

\[ \|\left[X(s) - X(t)\right]p\| \leq \|X(s)p - X(s_n)p\| + \|X(t_n)p - X(t)p\| + \|X(s_n)p - X(t_n)p\| \]

and, passing to the limit, we obtain

\[ \|X(s) - X(t)p\| \leq \limsup \|X(s_n) - X(t_n)p\| \leq \eta, \]

which completes the proof.

Now, we shall prove a theorem on the uniform continuity in Segal's sense of the mapping \( X \). Let \( g \) be a function defined on some interval \((-h_0,h_0)\), even and nondecreasing on \((0,h_0)\) and such that \( g(h) \to 0 \) as \( h \to 0 \).

**Theorem 2.** Let \( X : [a,b] \to \tilde{M} \) be a mapping such that

\[ \tau(e_{[g(h),\infty]}[|X(t + h) - X(t)|]) \leq f(h) \]

for some function \( f, f(h) \to 0 \) as \( h \to 0 \), each \( t, t + h \in [a,b] \), and let us assume that \( \sum_{n=1}^{\infty} r^n f(r^{-n}) < \infty \), \( \sum_{n=1}^{\infty} g(r^{-n}) < \infty \) for some integer \( r, r > 1 \). Then \( X \) is uniformly continuous in Segal's
sense.

If, moreover, \( \sum_{s=n+1}^{\infty} g(r^{-s}) \leq A g(r^{-n}) \) for some \( A \) and all \( n \), then there exists a positive constant \( C \) such that, for each \( \epsilon > 0 \), there is a projection \( p \) in \( M \) with \( \tau(p^2) < \epsilon \) and \( \delta > 0 \), such that \( \| [X(t + h) - X(t)] p \| \leq C g(h) \) for \( t, t + h \in [a, b] \) with \( |h| < \delta \).

**Proof.** To simplify the notation, take \( [a, b] = [0, 1] \). Let \( r > 1 \) be the integer in question and \( T = \{ k r^{-n} : k = 0, 1, \ldots r^{-n}, n = 1, 2, \ldots \} \) - the set of \( r \)-adic numbers in \( [0, 1] \).

Given \( \epsilon > 0 \), let \( N \) be such that \( \sum_{n=N}^{\infty} r^n f(r^{-n}) < \epsilon \). Put

\[
p = \bigwedge_{n=N}^{r-1} \bigwedge_{k=0}^n \epsilon \left[ g(r^{-n}) \right] \{ |X((k + 1) r^{-n}) - X(k r^{-n})| \}. \text{ Then } \tau(p^2) \leq \epsilon.
\]

\[
\leq \sum_{n=N}^{r-1} \sum_{k=0}^n \tau(e \left[ g(r^{-n}) \right]) \{ |X((k + 1) r^{-n}) - X(k r^{-n})| \} \leq \sum_{n=N}^{\infty} r^n f(r^{-n}) < \epsilon.
\]

For \( n \geq N \) and \( k = 0, 1, \ldots, r^{-n} - 1 \), we have

\[
\| X((k + 1) r^{-n}) - X(k r^{-n}) \| p \leq \epsilon.
\]

(2)

\[
\leq \sum_{n=N}^{r-1} \sum_{k=0}^n \tau(e \left[ g(r^{-n}) \right]) \{ |X((k + 1) r^{-n}) - X((kr + 1) r^{-n})| \} \leq \epsilon.
\]

Let \( t \in [kr^{-n}, (k+1)r^{-n}) \) be an \( r \)-adic number. Then \( t = kr^{-n} + \sum_{i=1}^{m} \theta_i r^{-(n+i)} \) where \( \theta_i = 0 \) or \( 1 \) or \( \ldots \) or \( r - 1 \) and \( m \) is some integer. Putting, for the sake of notation, \( \theta_0 = k \), we have

\[
[X(k r^{-n}) - X(t)] p = \sum_{s=0}^{m-1} X(\sum_{i=0}^{s} \theta_i r^{-(n+i)}) - X(\sum_{i=0}^{s+1} \theta_i r^{-(n+i)}) p
\]

and

\[
[X(\sum_{i=0}^{s} \theta_i r^{-(n+i)}) - X(\sum_{i=0}^{s+1} \theta_i r^{-(n+i)})] p =
\]

\[
\theta_{s+1} r^{-n} - \sum_{j=0}^{s} X(\sum_{i=0}^{j} \theta_i r^{-(n+i)} + j r^{-(n+s+1)}) - X(\sum_{i=0}^{s} \theta_i r^{-(n+i)} + (j+1) r^{-(n+s+1)}) p.
\]
Thus \([X(t) - X(\frac{kr-n}{r})]p\) belongs to \(M\) as a sum of operators belonging to \(M\), and, on account of (2),

\[
\|X(t) - X(\frac{kr-n}{r})\|p \leq \sum_{s=0}^{m-1} \|X(\sum_{i=0}^{s} \theta_{i} r^{-n+i}) - X(\sum_{i=0}^{s+1} \theta_{i} r^{-n+i})\|p \leq \\
\sum_{s=0}^{m-1} \sum_{j=0}^{s+1-l} \|X(\sum_{i=0}^{s} \theta_{i} r^{-n+i} + j r^{-n+i+1}) - X(\sum_{i=0}^{s} \theta_{i} r^{-n+i}) + (j+1) r^{-n+i+1})\|p \leq \\
\sum_{s=0}^{m-1} \sum_{j=0}^{s+1-l} r g(r^{-n+i+2}) = \sum_{s=1}^{m} r g(r^{-n+i+1}) \leq \\
r^2 \sum_{s=n+2}^{\infty} g(r^{-s}).
\]

Hence, for two \(r\)-adic numbers \(s, t \in [\frac{kr-n}{r}, (k+1)\frac{r-n}{r})\), \(k = 0, 1, \ldots, \frac{r-n}{r} - 1, \ n \geq N\), we have

\[(4) \quad \|X(t) - X(s)\|p \leq 2r^2 \sum_{s=n+2}^{\infty} g(r^{-s}).\]

Now, let \(\eta > 0\) be chosen arbitrarily. We find \(N_1\) such that

\[2r^2 \sum_{s=N_1+2}^{\infty} g(r^{-s}) < \eta/2\]

and \(N_2\) such that \(r g(r^{-(N_2+1)}) < \eta/2\). Let \(N_3 = \max(N, N_1, N_2)\) and take \(\delta = r^{-N_3}\). If \(s, t\) are \(r\)-adic numbers such that \(s < t, t - s < \delta\), then either

(i) \(s, t \in [\frac{k_0 r^{-N_3}}{r}, (k_0 + 1)\frac{r^{-N_3}}{r})\) for some \(k_0\) between 0 and \(r^{N_3} - 1\)

or

(ii) \(s \in [\frac{k_0 r^{-N_3}}{r}, (k_0 + 1)\frac{r^{-N_3}}{r}), \ t \in [(k_0 + 1)\frac{r^{-N_3}}{r}, (k_0 + 2)\frac{r^{-N_3}}{r})\)

for some \(k_0\) between 0 and \(r^{N_3} - 2\).

In case (i) we have, on account of (4),

\[
\|X(t) - X(s)\|p \leq 2r \sum_{s=N_3+2}^{\infty} g(r^{-s}) < \eta/2 < \eta
\]
and, for (ii), by virtue of (2) and (3), we get

\[ \| X(t) - X(s) \|_p \leq \| X(t) - X((k_0 + 1)r^{-N_3}) \|_p + \]

\[ + \| X((k_0 + 1)r^{-N_3}) - X(k_0r^{-N_3}) \|_p + \| X(s) - X(k_0r^{-N_3}) \|_p \leq \]

\[ \leq 2r^2 \sum_{s=N_3+2}^{\infty} g(r^{-S}) + rg(r^{-N_3+1}) < \eta/2 + \eta/2 = \eta. \]

It is easily seen that \( X \) is continuous in measure, thus the first statement follows from Proposition 1.

If we now assume that \( \sum_{s=n+1}^{\infty} g(r^{-S}) \leq A g(r^{-n}) \), then estimations (3) and (4) yield

\( (s,t) \in [kr^{-n}, (k+1)r^{-n}), \quad k = 0, 1, \ldots, r^{-n} - 1, \quad n \geq N. \) If \( s \in [kr^{-n}, (k+1)r^{-n}), \quad t \in [(k+1)r^{-n}, (k+2)r^{-n}]), \quad k = 0, 1, \ldots, r^{-n} - 2, \quad n \geq N, \) then, by virtue of (2) and (5), we get

\[ \| X(t) - X(s) \|_p \leq (2A r^2 + r) g(r^{-n+1}). \]

Put \( \delta = r^{-N} \) and let \( s, t \) (\( s < t \)) be arbitrary \( r \)-adic numbers with \( |s - t| < \delta \). We can find an \( n \geq N \) such that \( r^{-n+1} \leq \)

\( |s - t| < r^{-n}. \) There exists some \( k \) such that either

\( s \in [kr^{-n}, (k+1)r^{-n}) \) or \( s \in [kr^{-n}, (k+1)r^{-n}), \quad t \in [(k+1)r^{-n}, (k+2)r^{-n}]). \) Taking \( C = 2A r^2 + r \), we obtain from (6) and (7)

\[ \| X(t) - X(s) \|_p \leq C g(r^{-n+1}) \leq C g(|t - s|). \]

Now, it is easily seen that the uniform Segal continuity of \( X \), together with the density of the set of \( r \)-adic numbers in \([0,1]\), prove the second statement of the theorem.

**Corollary 3.** Let \( X, f \) and \( \alpha \) satisfy all the assumptions of Theorem 2 and, moreover, let \( g \) satisfy: \( g(s) + g(t) \leq g(t + s) \) for all sufficiently small positive numbers \( s, t \). If \( X \) is bounded in norm on \([a,b]\), then
\[ \|X(t+h) - X(t)\| \leq C g(h), \quad t, t+h \in [a, b]. \]

In particular, \( X \) is uniformly continuous in norm.

**Proof.** Choose a sequence \( \{\varepsilon_n\} \) of strictly positive numbers satisfying \( \sum_{n=1}^{\infty} \varepsilon_n < \infty \) and the corresponding sequences \( \{p_n\}, \{\delta_n\} \) as in the conclusion of Theorem 2. Put \( q_n = \bigwedge_{k=n}^{\infty} p_k \). Then \( q_n \uparrow 1 \) and
\[ \|X(t+h) - X(t)\|_{q_n} \leq C g(h) \quad \text{for} \quad |h| < \delta_n. \]
It can be seen that the above inequality is valid for arbitrary \( t, t+h \in [a, b] \). Indeed, it suffices to take a partition \( t = t_0 < t_1 < \ldots < t_m = t + h \) of diameter less than \( \delta_n \) and observe that \( \sum_{k=1}^{m} g(t_k - t_{k-1}) \leq g(h) \).

Now, for \( \xi \in \bigcup_{n=1}^{\infty} q_n H \) (where \( H \) is the space on which the algebra \( M \) acts), \( \|\xi\| \leq 1 \), we have
\[ \|X(t+h) - X(t)\|_{q_n} \leq C g(h). \]

Since \( \bigcup_{n=1}^{\infty} q_n H \) is a dense subspace of \( H \) and \( X \) is uniformly bounded, the result follows.

Our last corollary contains the "non-commutative Kolmogorov theorem on the continuity of trajectories of a stochastic process", as proved in [2].

**COROLLARY 4.** Let \( X : [a, b] \to M \) be a mapping such that

(i) \( \tau(|X(t+h) - X(t)|^\alpha) \leq L |h|^{1+\beta} \)

\[ \text{or} \]

(ii) \( \tau(|X(t+h) - X(t)|^\alpha) \leq L |h|/|\log |h||^{1+\beta}, \quad \alpha < \beta, \)

for some strictly positive constants \( L, \alpha, \beta \). Then \( X \) is uniformly continuous in Segal's sense and in case (i), for each \( 0 < \gamma < \beta/\alpha, \) \( X \) satisfies the "Hölder condition in Segal's sense with exponent \( \gamma \)", i.e., for each \( \varepsilon > 0, \) there are a projection \( p \) in \( M \) with \( \tau(p^+) < \varepsilon \) and some \( \delta > 0, \) such that \( \|X(t) - X(s)|^\alpha|p \| \leq C |t - s|^{\gamma} \) for some \( C > 0 \) and \( |t - s| < \delta. \)

The proof follows from the Tchebyshev inequality
\[ \tau(e^{[\lambda, \infty]}(|x|)) \leq \lambda^{-\alpha} \tau(|x|^\alpha) \quad \text{and from Theorem 2 with} \quad g(h) = |h|^{\gamma} \]
and \( f(h) = L |h|^{1+\beta-\alpha\gamma} \) in the first case and \( g(h) = |\log |h||^{-\theta} \)
1 < \Theta < \beta/\alpha, \text{ and } f(h) = L |h| / |\log |h||^{1+\beta-\alpha\Theta} \text{ in the other case.}

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Stable and semistable probabilities on groups and on vector spaces.

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Semistable and stable probabilities on vector spaces and - more recently - on locally compact groups were investigated by different authors (see e.g. [4,5,16,1,2] and the literature cited there; see also [17]). In this paper we continue the investigation of [4], [5]. Stability resp. semistability is considered as a property of a continuous convolution semigroup resp. of the corresponding generating distribution. A convolution semigroup \((\mu_t = \mathcal{C}p(tA))_{t \geq 0}\) on a locally compact group \(G\) is called semistable w.r.t. \(\tau \in \text{Aut}(G)\) and \(c \in (0,1)\) if \(\tau(\mu_t) = \mu_{ct}\) resp. \(\tau(A) = cA\) (more generally \(\tau(A) = cA + X\) for some \(X \in \mathfrak{g}\), where the elements of the Lie algebra \(\mathfrak{g}\) are identified with generating distributions of semigroups of point measures).

In §1 we study for given automorphisms the sets of [semi-] stable generating distributions and their properties, especially come structures, closedness in the weak* - topology and behaviour under mixing.

In §2 we study relations between semistable distributions on groups and on vector spaces. In [4], [5] it was shown that semistable generating distributions on Lie groups may be identified with generating distributions on vector spaces. This is true for any connected locally compact group admitting a contracting automorphism. On the other hand any semistable distribution on a metrizable group is concentrated on a measurable subgroup, on which the automorphism acts contracting.

We show that the metrizability condition is necessary, but not very restrictive, as a non metrizable group may be approximated by metrizable groups. If we do not suppose that \(\tau\) acts contracting it is an open problem if for connected groups any semistable distribution may be identified with a corresponding distribution on the Lie algebra.

We show by examples, illustrating semistable distributions on connected compact abelian groups, that it is not possible to reduce the problem to the case of finite dimensional groups.

In §3 we study limit theorems for semistable distributions. Originally operator - semistable and operator - selfdecomposable distributions on \(\mathbb{R}^d\) were defined (see [9,19]) als limits of distributions of normalized sums of independent random variables. Here we give for the
case of probabilities on groups some results connecting limit laws and semistable resp. selfdecomposable probabilities. Finally we give as an application of §1 a limit theorem for random products (with a random number of factors) on metrizable groups.

§1 - 3 are of expository character. There is a list of interesting open problems concerning semistable and decomposable measures on groups, which should be attacked in the sequel. I will mention some open questions:

1. Describe for certain classes of Lie groups $G$ completely the possible semistable generating distributions on $G$ and on the vector space $U_G$. (For example for groups of Heisenberg type, for semisimple groups etc.)

2. Describe the connection between semistable distributions on connected (infinite dimensional) groups and on the corresponding (infinite dimensional) Lie algebras. (This should especially be done for compact groups.)

3. Describe the connection between semistable measures and limit laws of normalized random products. To describe the domains of attraction it would be necessary to have a complete knowledge of the sets

$\{ \tau \in \text{Aut}(G) \mid A \text{ is semistable w.r.t. } \tau \}$

for a given generating distribution $A$. The description of the domain of attraction and of the possible norming automorphisms is still an open problem in the case of vector spaces.


§1 Some definitions and elementary facts.

In §1 we study generating distributions of continuous convolution semigroups $(\mu_t, t \geq 0, \mu_0 = \delta_e)$ (short: c.c.s.) on locally compact groups, which are (semi-) stable resp. (semi-) selfdecomposable with respect to an automorphism $\tau \in \text{Aut}(G)$ resp. to a multiplicative continuous group $(\tau_t)_{t>0} \subseteq \text{Aut}(G)$. The definitons of (semi-) stable resp. (semi-) self-decomposable distributions are natural generalizations of the usual stability concepts (s. [4,5]).

Let $G$ be a locally compact topological group. Let $\mathcal{D}(G)$ be the space of test functions and let $\mathcal{MO}(G) \subseteq \mathcal{D}'(G)$ be the cone of generating distributions of semigroups of probabilities on $G$. $\mathcal{M}^1(G)$ is the convolution semigroup of probabilities on $G$, $\ast$ denotes convolution. There is a 1-1-correspondence between distributions $A \in \mathcal{MO}(G)$ and convolution semigroups $(\mu_t, t \geq 0, \mu_0 = \delta_e) \subseteq \mathcal{M}^1(G)$ via $\langle A, f \rangle = \frac{d}{dt} \langle \mu_t, f \rangle|_{t=0}, f \in \mathcal{D}(G)$. Therefore we write

$$\mu_t := \exp(tA), t \geq 0.$$

The Lie Algebra $\mathfrak{g}$ is identified with $\mathcal{MO}(G) \cap (-\mathcal{MO}(G))$. (For details see e.g. [7,3]). The group of (topological) automorphisms of $G$ is denoted by $\text{Aut}(G)$. $\text{Aut}(G)$ is provided with the usual topology, but we frequently use the topology of pointwise convergence.

For $\tau \in \text{Aut}(G), f \in \mathcal{D}(G), F \in \mathcal{D}'(G)$ we define

$$\langle F, f \rangle := F(f), \tau(f) := f \circ \tau; \quad \tau(F) \text{ is defined via } \langle \tau(F), f \rangle := \langle F, \tau(f) \rangle, f \in \mathcal{D}(G).$$

1.1 Definition: Let $\tau \in \text{Aut}(G), c \in (0,1) \text{ be fixed. } A \in \mathcal{MO}(G)$ is called semistable w.r.t. $(\tau, c)$ if there exist $X \in \mathfrak{g}$, such that $\tau(A) = cA + X$.

$A$ is called semistable in the strict sense w.r.t. $(\tau, c)$ if $\tau(A) = cA$.

1.2 Definition: Let $\tau \in \text{Aut}(G) \text{ be fixed. } A \text{ is called semi-selfdecomposable w.r.t. } \tau \text{ if there exist } B \in \mathcal{MO}(G), \text{ such that } A = \tau(A) + B.$

1.3 Definition: Let $(\tau_t)_{t>0} \text{ be a group of automorphisms, such that } \tau_{t+s} = \tau_t \tau_s, t,s > 0, \text{ and such that } t \mapsto \tau_t \text{ is continuous w.r.t. pointwise convergence. } A \in \mathcal{MO}(G) \text{ is called stable w.r.t. } ((\tau_t)_{t>0}) \text{ if } \tau_t(A) = tA + X(t), t > 0, \text{ for some } X(t) \in \mathfrak{g}.$

$A \in \mathcal{MO}(G)$ is called stable in the strict sense w.r.t. $(\tau_t)$ if $\tau_t(A) = tA, t > 0$. 
1.4 Definition: Let $(\tau_t)$ be as before. $A \in MO(G)$ is called selfdecomposable w.r.t. $(\tau_t)_{t>0}$, if

$$A = \tau_t(A) + B(t),$$

for some $B(t) \in MO(G)$, $0 < t < 1$.

A semigroup $(\mu_t = \exp(tA))_{t \geq 0}$ is called (semi-) stable, (semi-) self-decomposable if the generating distribution $A$ has this property.

1.5 Definition: Now we denote the semistable distributions by

$$\mathcal{S}_s(\tau, c):= \{A \in MO(G) | \tau(A) = cA + X\},$$

$$\mathcal{S}_s(\tau, c):= \{A | \tau(A) = cA\}, \quad \mathcal{S}(\tau) := \bigcup_{c \in (0, 1)} \mathcal{S}_s(\tau, c).$$

Similar for $\alpha > 0$:

$$\mathcal{S}_t((\tau_t), \alpha) := \{A | \tau_t(A) = t^\alpha A + X(t), t > 0\},$$

$$\mathcal{S}_t((\tau_t), \alpha) = \bigcup_{\alpha > 0} \mathcal{S}_t((\tau_t), \alpha),$$

$$\mathcal{S}_t((\tau_t, \alpha)) = \{A | \tau_t(A) = t^\alpha A\}, \quad \mathcal{S}_t((\tau_t)) = \bigcup_{\alpha > 0} \mathcal{S}_t((\tau_t, \alpha)).$$

If $(\tau_t)_{t>0}$ is a continuous group, then for $\alpha > 0$ the $t = \tau_t^\alpha$ is a continuous group too. Obviously

$$\mathcal{S}_t((\tau_t, \alpha)) = \mathcal{S}_t((\tau_t^\alpha), 1).$$

We define finally

$$\mathcal{D}(\tau) = \{A | A = \tau(A) + B, \text{ for some } B \in MO(G)\},$$

$$\mathcal{D}((\tau_t)) = \{A | A = \tau_t(A) + B(t), t \in (0, 1), B(t) \in MO(G)\}.$$

Note: If $(\tau_t)_{t>0}$ is a continuous group $\leq \text{Aut}(G)$ and if $\alpha > 0$ then for any $A \in \mathcal{D}((\tau_t))$ we have $A \in \mathcal{D}((\tau_t^\alpha))$ too: We have

$$A = \tau_t(A) + B(t), \quad 0 < t < 1.$$ Therefore for $\alpha > 0$

$$A = \tau_t^\alpha(A) + B(t) \alpha = \tau_t^\alpha(A) + B(t) \alpha,$$

where $\alpha = \tau_t(A)$.

In the following propositions we collect some elementary facts concerning [semi-] stable and selfdecomposable generating distributions.

1.6 Proposition

(i) $\mathcal{S}_s(\tau, c) \subseteq \mathcal{S}(\tau, c)$

(ii) $\mathcal{S}_s((\tau_t), \alpha) \subseteq \mathcal{S}_s((\tau_t, \alpha))$

(iii) If $A \in \mathcal{S}_s(\tau, c) \bigcup \mathcal{S}_s((\tau_t), \alpha))$ and $\bar{A} = A$, then

$$A \in \mathcal{S}_s(\tau, c) \bigcup \mathcal{S}_s((\tau_t), \alpha))$$

(where $<\bar{A}, f> := <A, f>$, $\bar{f}(x) := f(x^{-1})$)

(iv) $\mathcal{S}_s((\tau_t), \alpha) \subseteq \mathcal{S}(\tau_{t_0}, t_0^\alpha)$ for any $t_0 \in (0, 1)$
(v) $\mathcal{Y}(\tau, c) \subseteq \mathcal{I}(\tau)$  
(vi) $\mathcal{I}(\tau_t) \subseteq \mathcal{D}(\tau_t)$  
(vii) $\mathcal{D}(\tau_t) \subseteq \mathcal{I}(\tau_{t_0})$ for any $t_0 \in (0,1)$

(i) - (iv), (vii) are obvious.

(v) Assume $\tau(A) = c \cdot A + X$, $X \in \mathcal{G}$, $0 < c < 1$.
Then $A = cA + (1-c)A = \tau(A) + [(1-c)A - X]$.

(vi) Similar. If $\tau_t(A) = t^\alpha A + X(t)$, $X(t) \in \mathcal{G}$, $0 < t < 1$, $\alpha > 0$,
then $A = t^\alpha A + (1-t^\alpha)A = \tau_t(A) + [(1-t^\alpha)A - X(t)]$.

1.7 Proposition (Cone structures).
For fixed $c \in (0,1)$, $\tau \in \text{Aut}(G)$, $(\tau_t)_{t>0} \subseteq \text{Aut}(G)$, $\alpha > 0$ we have
(i) $\mathcal{Y}_S(\tau, c)$, $\mathcal{Y}(\tau, c)$ are convex cones  
(ii) $\mathcal{I}_S((\tau_t), \alpha)$, $\mathcal{I}((\tau_t), \alpha)$ are convex cones  
(iii) $\mathcal{D}(\tau)$ and $\mathcal{D}((\tau_t))$ are convex cones  
(iv) $\mathcal{I}(\tau)$, $\mathcal{I}((\tau_t))$ are not convex,
co($\mathcal{I}(\tau)$) resp. co($\mathcal{I}((\tau_t))$) (the convex hull of $\mathcal{I}(\tau)$ resp. $\mathcal{I}((\tau_t))$) are contained in $\mathcal{I}(\tau)$ resp. $\mathcal{D}((\tau_t))$.

Assume $A, B \in \mathcal{Y}(\tau, c)$. Then obviously for any $\lambda > 0$

$\lambda A, \lambda B \in \mathcal{Y}(\tau, c)$. So it is sufficient to show that $A + B \in \mathcal{Y}(\tau, c)$ too. We have $\tau(A) = cA + X_A$, $\tau(B) = cB + X_B$.

$\tau(A + B) = \tau(A) + \tau(B) = cA + X_A + cB + X_B = c(A + B) + X_A + X_B$.

The cones $\mathcal{Y}_S(\tau, c)$, $\mathcal{I}((\tau_t), \alpha)$, $\mathcal{I}_S((\tau_t), \alpha)$ are treated in an analogous manner. (iii) is obvious.

(iv) It is sufficient to show that

$\mathcal{I}(\tau) + \mathcal{Y}(\tau) \subseteq \mathcal{I}(\tau)$ [resp. $\mathcal{I}((\tau_t)) + \mathcal{I}((\tau_t)) \subseteq \mathcal{D}((\tau_t))$].

Assume $\tau(A) = cA + X$, $\tau(B) = dB + Y$, $c,d \in (0,1)$, $X,Y \in \mathcal{G}$. Therefore $A = \tau(A) + (1-c) A - X, B = \tau(B) + (1-d) B - Y$, whence $A + B = \tau(A + B) + [(1-c) A + (1-d) B - X - Y]$.

The case $\mathcal{I}((\tau_t))$ is treated in a similar manner. It is easily seen, that in general $\mathcal{I}(\tau)$ and $\mathcal{I}((\tau_t))$ are not convex. \[\]
1.8 Proposition  The subsets $\mathcal{I}_s(\tau, c), \mathcal{I}(\tau, c), \mathcal{I}_s((\tau_t), a), \mathcal{I}((\tau_t), a)$, $\mathcal{J}(\tau), \mathcal{J}((\tau_t)) \subseteq \mathcal{M}(G)$ are closed in the $C(\mathcal{D}'(G), \mathcal{D}(G))$-topology restricted to $\mathcal{M}(G)$.

Assume that $(A_a)_{a \in I} \subseteq \mathcal{M}(G)$ is a net, $C(\mathcal{D}', \mathcal{D})$-convergent to $A \in \mathcal{M}(G)$. Assume further $\tau(A_a) = c_A + X_a, X_a \in \mathcal{G}$.

As $\tau(A_a) \to \tau(A), c_A \to c_A$, we have $\tau(A) = \tau(A_0) + X \quad (X = \tau(A) - c_A = \lim\limits_{\alpha} X_a \in \mathcal{G}),$ i.e. $A \in \mathcal{I}(\tau, c)$.

Now assume that $A_a \underset{a \in I}{\longrightarrow} A, A \in \mathcal{J}(\tau), A \in \mathcal{M}(G)$.

Then $A_a = \tau(A_a) + B_a, B_a \in \mathcal{M}(G), A_a \to A, \tau(A_a) \to \tau(A),$ therefore $B_a \to A - \tau(A) =: B,$ and we obtain $A = \tau(A) + B$.

Hence $A \in \mathcal{J}(\tau)$.

The remaining assertions are proved in a similar manner. \[]

1.9 Proposition  Let $\tau \in \text{Aut}(G)$ resp. $(\tau_t)_{t > 0} \subseteq \text{Aut}(G)$ be fixed and assume $0 < a \leq b < 1$ resp. $0 < u \leq v < \infty$. Then the following subsets of $\mathcal{M}(G)$ are $C(\mathcal{D}', \mathcal{D})$-closed:

(i) $\bigcup_{a \leq c \leq b} \mathcal{I}(\tau, c), \quad$ (ii) $\bigcup_{u \leq a \leq v} \mathcal{I}_s((\tau_t), a)$

If $G$ is a connected Lie group and $K \subseteq \text{Aut}(G)$ is compact, then the following sets are $C(\mathcal{D}', \mathcal{D})$-closed in $\mathcal{M}(G)$:

(iii) $\bigcup_{a \leq c \leq b, \tau \in K} \mathcal{I}(\tau, c), \quad$ (iv) $\bigcup_{\tau \in K} \mathcal{J}(\tau)$

Proof: (i) Let $(A_a)_{a \in I}$ be a $C(\mathcal{D}', \mathcal{D})$-convergent net in $\mathcal{I}(\tau, c)$. If $A_a \underset{a \in I}{\longrightarrow} A \in \mathcal{M}(G)$ there exists a net $(c_a)_{a \in I} \subseteq [a, b]$ such that $\tau(A_a) = c_a A_a + X_a$.

We have $A_a \to A$, hence $\tau(A_a) \to \tau(A)$. W.l.o.g. we may assume that $c_a \to c_o \in [a, b]$. Hence $X_a \to X_o := \tau(A) - c_o A$, therefore $\tau(A) = c_o A + X_o$.

(ii) Is proved in an analogous manner.

(iii) If $G$ is a connected Lie group, then $\text{Aut}(G)$ is a Lie group and for $f \in \mathfrak{g}(G)$ the set $\{ f \circ \tau \mid \tau \in K \}$ is compact. $\mathcal{D}'$ is a Fréchet-space, $(A, f) \mapsto <A, f> : \mathcal{D}'(G) \times \mathcal{D}(G) \to \mathbb{R}$ is equicontinuous on compact
subsets of $\mathcal{D}(G)$. Assume now $(A_n)_{n \in \mathbb{N}}$ to be a sequence in $\bigcup_{\alpha \in \mathbb{R}} \bigcup_{\tau \in K} \mathcal{Y}(\tau, c)$, 
$\tau_n(A_n) = c_n A_n + X_n$ and assume w.l.o.g. $\tau_n \to \tau_0$, $c_n \to c_0$, $A_n \to A_0$.
Then for $f \in \mathcal{D}(G)$ 
$< \tau_n(A_n), f > = < A_n, f \circ \tau_n > = < A_0, f \circ \tau_0 > = < \tau_0(A), f >$.

On the other hand 
$< c_n A_n + X_n, f > \to < c_0 A_0 + X_0, f >$,
where $X_0 = \lim X_n = A - \tau_0(A)$.

The case (iv) is treated in a similar manner. 

1.10 Corollary $\mathcal{L}_s(\tau, c), \mathcal{L}(\tau, c), \mathcal{L}_s((\tau_t), a), \mathcal{L}_t((\tau_t), a), 
\mathcal{L}(\tau)$ and $\mathcal{D}((\tau_t))$ are closed w.r.t. mixing: Assume that $(\Omega, \Sigma, P)$ 
is a probability space and that $\phi : \omega \to A_\omega$ is a $\mathcal{L}(\mathcal{D}', \mathcal{D}) - P$-integrable
map $\Omega \to \mathcal{M}(G)$. Define $A_0 := \int_\Omega A_\omega \, dP(\omega)$ as $\mathcal{L}(\mathcal{D}', \mathcal{D})$ convergent integral.

If $A_\omega \in \mathcal{L}_s(\tau, c)$ [resp. $\mathcal{L}(\tau, c), \mathcal{L}_t((\tau_t), a), \mathcal{L}_t((\tau_t), a), \mathcal{L}(\tau), \mathcal{D}(\tau), \mathcal{D}((\tau_t))]$
then $A_0 \in \mathcal{L}_s(\tau, c)$ [resp. $\mathcal{L}(\tau, c), \mathcal{L}_t((\tau_t), a), \mathcal{L}_t((\tau_t), a), \mathcal{L}(\tau), \mathcal{D}(\tau), \mathcal{D}((\tau_t))]$ too.

Follows immediately from 1.7 and 1.8 resp. [3] I. 1.3 \[3\]. 

For details about mixing of generating distributions see e.g. [3, 6].

In the next proposition we show that it is possible to generate [semi]-
stable or [semi]-selfdecomposable distributions via subordination.

Recall that $M_1(\mathbb{R})$ is a convolution semigroup $\leq M_1(\mathbb{R})$. The generating
distributions $F$ of continuous convolution semigroups $(\nu_t)_{t \geq 0} \subseteq M_1(\mathbb{R})$
have the following form $\alpha \in \mathbb{R}_+, \eta_F$ a Lévy - measure on $\mathbb{R}_+$,
$< f, F > = \alpha f'(0) + \int (f(x) - f(0)) \, d\eta_F(x)$. We write shortly 
$(0, \infty)$
$F = (\alpha \frac{d}{dx}, \eta_F)$. The semistable, stable and selfdecomposable distribu-
tions on $\mathbb{R}_+$ are well known (s. e.g. [0] and the literature cited there). 
For any $b > 0$ call $\delta_b$ the automorphism $x \mapsto bx$, $x \in \mathbb{R}$. We have 
$F = (\alpha \frac{d}{dx}, \eta_F) \in \mathcal{L}(\delta_b, u)$ [resp.$\in \mathcal{L}_s(\delta_b, u)$] 

iff $\delta_b(F) = (b \cdot \alpha \frac{d}{dx}, \delta_b(\eta_F)) = u \cdot F + \beta \cdot \frac{d}{dx}$, for some $\beta > 0$,
i.e. $\delta_b(\eta_F) = u \cdot \eta_F$ and $ba = c \cdot \alpha + \beta$. [If 
$F \in \mathcal{L}_s(\delta_b, c)$ $\delta_b(\eta_F) = c \cdot \eta_F$ and $ba = ca$, hence $a = 0$ or $b = c$ ].
$F \in \mathcal{L}_t((\delta_t), a), \mathcal{L}_s((\delta_t), a)$ resp. $\mathcal{L}(\delta_b)$ are easily 
characterized in a similar way.
1.11 Proposition

Assume that $A$ is strictly semistable, $A \in \mathcal{S}(\tau, c)$. Then for $F, G \in \mathcal{M}(\mathbb{R}_+)$, 
$F \in \mathcal{S}(\delta_c, u)$, $G \in \mathcal{M}(\delta_c)$ with $\exp(tF) =: \eta_F$, $\exp(tG) =: \gamma_t$, $t \geq 0$,
we obtain: If $U$ resp. $V$ are the generating distributions of the subordinated semigroups in $\mathcal{M}(G)$

$$
\left( \int_{(0, \infty)} \exp(sA) d\eta_F(s) \right)_{t \geq 0} \text{ resp. } \left( \int_{(0, \infty)} \exp(sA) d\gamma_t(s) \right)_{t \geq 0},
$$

then $U \in \mathcal{S}(\tau, u)$ resp. $V \in \mathcal{M}(\tau)$.

If $F \in \mathcal{S}(\delta_c, u) \setminus \mathcal{S}(\delta_c, u)$, then in general $U \notin \mathcal{S}(\tau)$, but $U \in \mathcal{M}(\tau)$.

Proof: Assume $A \in \mathcal{S}(\tau, c)$, $\tau(A) = cA$ resp. $\tau(\exp tA) = \exp(t\tau(A))$, $t \geq 0$.
Assume further $F = (\alpha \frac{d}{d\alpha}, \eta_F) \in \mathcal{S}(\delta_c, u)$ and 
$G = (\beta \frac{d}{d\beta}, \eta_G) \in \mathcal{M}(\delta_c)$.

Then $\delta_c(\eta_F) = u \cdot \eta_F$, $\alpha = u$, resp. $\delta_c(\eta_G) = \eta_G$, $t \geq 0$,
and $\eta_G = \delta_c(\eta_G) + \eta_B$ for some Lévy measure $\eta_B$.
Let $U$ be the generating distribution in $\mathcal{M}(G)$ with 

$$
\int_{(0, \infty)} \exp(sA) d\eta_F(s) = \exp(tU), \quad t \geq 0.
$$

Therefore $U = \alpha A + \int_{(0, \infty)} (\mathcal{E}(\exp(tA)) - \epsilon_e) d\eta_F(t)$.

Hence $\tau(U) = \alpha \tau(A) + \int_{(0, \infty)} (\mathcal{E}(\exp(t\tau(A)) - \epsilon_e) d\eta_F(t)$

$$
= \alpha \cdot cA + \int_{(0, \infty)} (\mathcal{E}(\exp(t\tau(A)) - \epsilon_e) d\eta_F(t) =
$$

$$
= \alpha \cdot c \cdot A + \int_{(0, \infty)} (\mathcal{E}(\exp(sA) - \epsilon_e) d\eta_F(s/c) =
$$

$$
= \alpha \cdot c \cdot A + \int_{(0, \infty)} (\mathcal{E}(\exp(sA) - \epsilon_e) d\delta_c(\eta_F(s)) =
$$

$$
= \alpha \cdot uA + u \cdot \int_{(0, \infty)} (\mathcal{E}(\exp(sA) - \epsilon_e) d\eta_F(s) =
$$

$$
= u \cdot U.
$$

In a similar manner we obtain:

If $\delta_c(F) = uF + \beta \frac{d}{d\beta}$, $\beta > 0$, then $\tau(U) = \beta A + u \cdot U$.

Hence, if $A \notin \mathcal{S}$, and $A + \delta \cdot U + Y$ for $\delta \in \mathbb{R}$, $Y \in \mathcal{S}$, then $U \notin \mathcal{S}(\tau)$. 
To proof the second assertion put for $f \in \mathfrak{D}(G)$

$$\phi : (0, \infty) \ni t \mapsto \langle \exp(tA), f \rangle \in \mathbb{R}.$$  

From $\phi(\delta_c(t)) = \langle \exp(ctA), f \rangle = \langle \tau(\exp(tA)), f \rangle = \langle \exp(tA), f \circ \tau \rangle,$

we obtain for $V : \langle V, f \rangle = \langle F, \phi \rangle,$ hence

$$\langle \tau(V), f \circ \tau \rangle = \langle F, \phi \circ \delta_c \rangle = \delta_c(F), \phi \rangle = \langle F, \phi \rangle - \langle B, \phi \rangle = \langle V, f \rangle - \langle B, \phi \rangle$$  

(from $F = \delta_c(F) + B$).

Hence $V = \tau(V) + W$, where $\langle W, f \rangle = \langle B, \phi \rangle.$  

The following result is proved in a similar manner:

**1.12 Proposition**

Let $(\tau_t)_{t \geq 0}$ be a continuous group in Aut($G$). Let $A \in \mathfrak{M}(G)$ be a strictly stable generating distribution, $A \in \mathfrak{S}(\alpha).$ Assume further that $(\tau_t)$ resp. $(\tau^t)$ are continuous convolution semigroups in $M^1(\mathbb{R}_+)$ with generating distributions $F$ resp. $G$.

If $F \in \mathfrak{S}(\delta_t, \beta), \text{[resp. } \mathfrak{S}(\delta_t^0, u)]$, then for the generating distribution $U$ of $\int_{(0, \infty)} \exp(sA) \, d\eta_t(s))_{t \geq 0}$ we have:

$$U \in \mathfrak{S}(\tau_t, \beta \alpha) \text{[resp. } U \in \mathfrak{S}(\tau_t^1/\alpha, u)]$$

If $G \in \mathfrak{D}(\delta_t)$ then for the generating distribution $V$ of $\int_{(0, \infty)} \exp(sA) \, d\xi_t(s))_{t \geq 0}$ we have $V \in \mathfrak{D}(\tau_t)$.

Proof: Put for $f \in \mathfrak{D}(G) \phi : (0, \infty) \ni t \mapsto \langle \exp(tA), f \rangle.$

We have $\phi \circ \delta_t(s) = \langle \exp(sA), f \rangle = \langle \tau_{t/\alpha}(\exp(sA)), f \rangle = \langle \exp(sA), f \circ \tau_{t/\alpha} \rangle.$

Therefore, as $\langle U, f \rangle = \langle F, \phi \rangle$, we have

$$\langle \tau_{t/\alpha}(U), f \rangle = \langle F, \phi \circ \delta_t \rangle = \delta_t(F), \phi \rangle = t^{\beta} \langle F, \phi \rangle = t^{\beta} U.$$

Hence $\tau_t(U) = t^{\beta} U.$

If $\delta_t(F) = u \cdot F$, then we have

$$\phi \circ \delta_t^0(s) = \langle \exp(sA), f \circ \tau_{t/\alpha}^1 \rangle.$$

Therefore $\langle \tau_{t/\alpha}^1(U), f \rangle = \langle \delta_t^0(F), \phi \rangle = u \langle U, f \rangle.$
Then 

Assume now \( G \in \mathcal{D}(\delta_t) \), i.e. 

\[ G = \delta_t(G) + B(t), \quad 0 < t < 1, \quad B(t) \in \mathcal{M}_0(\mathbb{R}_+) \] 

Then 

\[ < \tau_t^1/\alpha(V), f > = < \delta_t(G), \phi > = \]

\[ = < G, \phi > - < B(t), \phi > = < V, f > - < W(t), f > \]

(with \( < W(t), f > = < B(t), \phi > \)).

Therefore we obtain with \( \overset{\cdot}{W}(t) := W(t^a) \):

\[ V = \tau_t(V) + \overset{\cdot}{W}(t), \quad 0 < t < 1. \]

\[ \square \]

§ 2 Contracting automorphisms

To make the paper more self-contained we give a short survey on known results. Let \( G \) be a locally compact group and \( \tau \in \text{Aut}(G) \) [resp. let \( (\tau_t)_{t>0} \) be a multiplicative group in \( \text{Aut}(G) \)].

\( \tau \) resp. \( (\tau_t) \) is called contracting in \( x \in G \)

if \( \tau^k(x) \to e \) resp. \( \tau_t(x) \to e \). \( \tau \) resp. \( (\tau_t) \) is contracting on a set \( B \subseteq G \) if \( \tau \) resp. \( (\tau_t) \) acts contracting in \( x \) for \( x \in B \).

\( (2.1) \) The connected component of \( e \) \( G_0 \) is a \( \tau - [(\tau_t)] \)-invariant subgroup. If \( \tau[(\tau_t)] \) is contracting on \( G \), then \( G_0 \) is a nilpotent and simply connected Lie group and \( G \) splits \( G = G_0 \oplus H \) ([18]).

\( (2.2) \) If \( \pi : G \to G/G_0 \) is the canonical projection, then there exists \( \tilde{\tau} \in \text{Aut}(G/G_0) \) such that \( \pi \circ \tau = \tilde{\tau} \circ \pi \). If \( \tau \) acts contracting on \( G \), then \( \tilde{\tau} \) acts contracting on \( G/G_0 \) (see [5] resp. [18]).

\( (2.3) \) If \( A \in \mathcal{M}_0(G) \) is [semi-] stable resp. [semi-] selfdecomposable on \( G \) w.r.t. \( \tau \) resp. \( (\tau_t) \), then \( \pi(A) \) is [semi-] stable resp. [semi-] self-decomposable w.r.t. \( \tilde{\tau} \) resp. \( (\tilde{\tau}_t) \) (see [5] 3.3).

\( (2.4) \) If \( A \) is stable on \( G \), \( A \in \mathcal{S}((\tau_t), \alpha) \), then \( (\alpha_t = \text{Exp}(tA))_{t>0} \) is concentrated on \( G_0 \) (see [5] 3.3).

\( (2.5) \) If \( G \) is a Lie group and if \( A \in \mathcal{M}_0(G) \) is semistable, then \( (\alpha_t = \text{Exp}(tA)) \) is concentrated on \( G_0 \) (see [5] 1.9).

\( (2.6) \) If \( G \) is a nilpotent, connected, simply connected Lie group, then
there exists a 1-1 correspondence between $M\mathcal{O}(G)$ and $M\mathcal{O}(\mathfrak{g})$ - ($\mathfrak{g}$ denotes the Lie algebra of $G$) via $\langle A, f \circ \exp \rangle = \langle A, f \rangle$, $A \in M\mathcal{O}(G)$, $f \in \mathfrak{g}(G)$. For $\tau \in \text{Aut}(G)$ we denote $dt = d\tau$. We obtain:

A $\in \mathcal{Y}(\tau, c)$ iff $A \in \mathcal{Y}(\mathfrak{g}, c)$ resp. $A \in \mathcal{Y}_t((\mathfrak{g}, t), \alpha)$ iff $A \in \mathcal{Y}_t((\mathfrak{g}, t), \alpha)$; $A \in \mathcal{Y}_s((\mathfrak{g}, t), \alpha)$ iff $A \in \mathcal{Y}_s((\mathfrak{g}, t), \alpha)$; $A \in \mathcal{D}(\tau)$ iff $A \in \mathcal{D}(\mathfrak{g})$; $A \in \mathcal{D}(\mathfrak{g}, t)$ iff $A \in \mathcal{D}(\mathfrak{g}, t)$ (see [5] 2.1, see also [4]).

This enables us to describe completely the classes of generating distributions mentioned above: $\mathcal{Y}$ resp. $\mathcal{Y}_t$ are special vector space automorphisms, therefore $\mathcal{Y}(\mathfrak{g}, c)$ resp. $\mathcal{Y}_t((\mathfrak{g}, t), \alpha)$ etc. are classes of operator-semistable resp. operator-stable etc. distributions.

The form of these generating distributions is completely known: For stable distributions see M. Sharpe [15], Hudson, Mason [8]; for semistable distributions see R. Jajte [9], A. Luczak [13, 14]; for self-decomposable distributions see R. Jajte [10], K. Urbanik [19]. See also Z.J. Jurek [11, 12].

(2.7) If $G$ is metrizable and if $A \in \mathcal{Y}_s((\mathfrak{g}, t), \alpha)$, then there exists a measurable subset $B \in G$ on which $\tau$ resp. $(\mathfrak{g}, t)$ acts contracting, such that $\exp(tA)$ is concentrated on $B$ (see [5] 2.3).

(2.8) If $G$ is a Lie group, then for any $A \in \mathcal{Y}_s((\mathfrak{g}, t), \alpha)$ there exists (i) $Y \in \mathcal{Y}$, (ii) a $\tau$-resp. $(\mathfrak{g}, t)$-invariant analytic subgroup $G_1 \leq G_0$, such that $\exp(t(A - Y))$ is concentrated on $G_1$, and such that $\tau$ resp. $(\mathfrak{g}, t)$ acts contracting on $G_1$.

(See [5] 2.8). Especially $G_1$ is nilpotent and simply connected. Therefore the reduction procedure (2.6) enables us to describe completely the structure of [semi-] stable distributions on Lie groups. For the Heisenberg group a complete description of the set of all stable generating distributions and measures is given by T. Drisch, L. Gallardo [1].

In the following theorem and examples we show that the metrizability condition in (2.7) is not very restrictive:

**2.9 Theorem** Let $G$ be a locally compact group (in general not metrizable). Assume that $\tau \in \text{Aut}(G)$ and $A \in M\mathcal{O}(G)$ are such that $A \in \mathcal{Y}_s(\mathfrak{g})$.

Then there exists an open $\mathcal{G}$-compact, $\tau$-invariant subgroup $G_1$ supporting $\exp(tA)$, such that

$G_1$ is a projective limit of metrizable groups $G_1/K^*$ with $\tau$-invariant
compact subgroups $K^* \triangleleft G_1$.

Therefore: For any neighbourhood $U \in \mathcal{U}(e)$ there exists a compact subgroup $K^* \triangleleft U$ and a measurable $\tau$-invariant subset $\Gamma_U$, on which $\mathcal{E}xp(tA)$ is concentrated, such that for $x \in \Gamma_U$ there exists $N_x \in \mathbb{N}$ with $\tau^k(x) \in U$ for $k \geq N_x$.

Proof: We know by [5] 3.1 that there exists an open $\mathcal{G}$-compact $\tau$-invariant subgroup $G_1$ supporting $A$.

Put $\mathcal{L} := \{ K \mid K$ compact normal subgroup $\subseteq G_1, \text{such that } G_1/K$ is metrizable $\}$.

For fixed $K \in \mathcal{L}$ we put $K^* := \bigcap_{k \in \mathbb{Z}} \tau^k(K)$.

Obviously $K^*$ is a compact normal subgroup and $K^* \subseteq \tau^k(K)$. Furthermore $G_1/\tau^k(K)$ is metrizable as $G_1/K = G_1/\tau^k(K)$ for any $k \in \mathbb{Z}$.

It is easily shown that $G_1/K^* = G_1/\cap_{k \in \mathbb{Z}} \tau^k(K)$ is metrizable.

We obtain: $G_1 = \lim_{k \to \infty} G_1/K^*$. Put $\mathcal{L}^* := \{ K^* \mid K \in \mathcal{L} \}$.

For $K^* \in \mathcal{L}^*$ let $\pi_{K^*}$ be the canonical projection $\pi_{K^*} : G_1 \to G_1/K^*$, and define $\tau^* \in Aut(G_1/K^*)$ by $\tau^* \pi_{K^*} = \pi_{\tau^k(K)}$.

Obviously $\pi_{K^*}(A) \in \mathcal{S}_{\mathcal{S}}(K^*)$, therefore by (2.7)

$\mathcal{E}xp(t K^* A) = \pi_{K^*}(\mathcal{E}xp(t A))$ is concentrated on a subset $B_{K^*} \subseteq G_1/K^*$, on which $\tau^*$ acts contracting.

Put $\Gamma_U := \pi^{-1}_{K^*}(B_{K^*})$. Then the assertions are immediately proved.

2.10 Example We know (see [5] 4.2) that there exist compact groups $K$, which admit an automorphism $\varphi \in Aut(K)$ and semistable distributions $B \in \mathcal{S}_{\mathcal{S}}(\tau, c)$. (In [5] $K$ is a solenoidal group and $B$ is the image via $\exp$ of a semistable distribution on $\mathbb{R}$). $\varphi$ is contracting on a dense subgroup $\Gamma \triangleleft K$, on which $B$ is concentrated.

Put $G := K^I$ (for some index-set $I$) endowed with the product topology, and define $\tau \in Aut(G)$ via

$(\tau(\omega))(k) := \varphi(\varphi(k)), k \in \mathbb{Z}$ (for $\varphi : I \to K$). We define further $A \in \mathcal{M}(G)$ via $\mathcal{E}xp(t A) =: \mu_t = \bigotimes_{k \in I} \mathcal{E}xp(tB)$.

Obviously $\tau(\mathcal{E}xp(tA)) = \bigotimes_{k \in I} \varphi(\mathcal{E}xp(tB)) = \bigotimes_{k \in I} \mathcal{E}xp(tcB) = \mathcal{E}xp(tcA)$, thus $\tau(A) = cA$.

$\tau$ acts contracting on $\bigotimes_{\Gamma}$, but, if $I$ is non-countable,
\[ \rho \] has outer measure \( \mu^*_t(\rho) = 1 \) and inner measure \( \mu^*_{t}(\rho) = 0 \).

So in the non metrizable case (2.7) is not valid.

**Examples** We show that this can even happen in the case of arcwise connected abelian groups (see 2.13):

2.11.a Lemma Assume that \( G \) is a locally compact group, \( \tau \in \text{Aut}(G) \), \( B \in \mathcal{MO}(G) \), \( c \in (0,1) \). Assume that for \( f \in \mathcal{D}(G) \)

\[
\sum_{k=-\infty}^{\infty} c^{-k} \cdot (B) \cdot f \quad \text{is absolutely convergent.}
\]

Then \( A : f \mapsto \langle A, f \rangle := \sum_{k=-\infty}^{\infty} c^{-k} \cdot (B) \cdot f \cdot \tau^k \) belongs to \( \mathcal{F}_s(\tau, c) \).

\[ \square \] Obvious. \( \square \)

2.11.b There exist nontrivial semistable distributions on the infinite dimensional torus \( T^{\mathbb{R}} \) more general, on any group \( K^{\mathbb{R}} \), for a nontrivial compact group \( K \): Let \( K \neq \{ e \} \) be a compact group (e.g. \( K = T \)). Define \( \rho : \mathcal{H} := K^{\mathbb{R}} / K \) to be the shift

\[
\rho(\phi)(k) := \phi(k+1), \quad k \in \mathbb{Z}.
\]

Then \( \rho \) acts contracting on

\[
H := \{ \phi \mid \text{card } \{ k \mid \phi(k) \neq e \} < \infty \}.
\]

For any \( n_1 < n_2 \in \mathbb{Z} \), for any generating distribution

\[
B \in \mathcal{MO}(T^{n_2} / K)
\]

we define \( B \in \mathcal{MO}(\mathcal{H}) \) via

\[
\langle B, f \rangle := \langle B, f \circ i_{n_1, n_2} \rangle
\]

(where \( i_{n_1, n_2} : T^{n_2} / K \to K^{\mathbb{R}} \) is the canonical embedding).

For any \( f \in \mathcal{D}(\mathcal{H}) \) obviously \( \sum_{k=-\infty}^{\infty} c^{-k} \cdot (B) \cdot f \cdot \rho^k \) converges (indeed the entries are zero, except a finite number).

Therefore \( A = \sum_{k=-\infty}^{\infty} c^{-k} \cdot \rho^k \) is a semistable distribution on \( \mathcal{H} \).

2.11.c If \( K \) is finite, then \( K^n \) does not admit automorphisms which act contracting on a dense subset, but \( K^{\mathbb{Z}} = \mathcal{H} \) does.

If \( K = T \) (the one dimensional torus) then \( T^n(n > 1) \) admits automorphisms, which act contracting on dense subsets: As \( \text{Aut}(T^n) \) is isomorphic to \( \text{SL}(n, \mathbb{Z}) \) any automorphism \( \tau \) of \( T^n \) is induced by an automorphism \( \phi \) of the Lie algebra \( \mathbb{R}^n \), such that \( \phi \) acts contracting on a subspace \( V \), the image of which under \( \exp : \mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n = T^n \) is dense, and such that \( \exp : V \to H := \exp(V) \) is bijective.
2.11.d The automorphism \( \mathfrak{g} \) on \( \mathbb{T}^{\mathbb{Z}} \) induced by the shift (acting contracting on the dense subgroup \( H = \{ \varphi \mid \text{card} \{ k \mid \varphi(k) \neq e \} < \infty \} \),

is of quite different character: The group \( \mathbb{T}^{\mathbb{Z}} \) is arcwise connected, the exponential map \( \exp : \mathbb{R}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}} \) is not injective. There exists a subspace \( \mathfrak{g} \) of the Lie algebra \( \mathfrak{g} = \mathbb{R}^{\mathbb{Z}} \) and an automorphism \( \mathfrak{j} \) on \( \mathfrak{g} = \mathbb{R}^{\mathbb{Z}} \), which acts contracting on \( \mathfrak{g} \), such that \( \mathfrak{g} \) is induced by \( \mathfrak{j} \) and such that \( \exp(\mathfrak{g}) \neq H \). But it is not possible to describe \( \mathfrak{j} \) by its action on finite dimensional subspaces. Consequently the possible semistable distributions in 2.11.c and 2.11.b differ in a characteristic manner:

2.12.a Assume \( \mathfrak{X} = \mathbb{T}^{\mathbb{Z}} \), \( n \in \mathbb{N} \), \( \varphi \in \text{SL}(n, \mathbb{Z}) \) acting contracting on a dense subgroup \( \mathbb{H} = \{ \varphi \mid \text{card} \{ k \mid \varphi(k) \neq e \} < \infty \} \), and assume \( V \subseteq \mathbb{R}^n \) to be a linear subspace such that \( \exp(V) = \mathbb{H} \) and such that \( \varphi \) acts contracting on \( V \). Then there exists a semistable generating distribution \( \overline{A} \in \mathcal{M}_0(V) \) (in the sense of Jajte), such that \( \varphi(\overline{A}) = c_0\overline{A} \), and \( \overline{A} \) is uniquely determined by 2.6. Therefore \( A := \exp(\overline{A}) \in \mathcal{M}_0(\mathfrak{X}) \) is semistable w.r.t. \( (\varphi, c) \).

(As \( \exp : \mathbb{V} = \mathbb{R}^n = \mathbb{H} \) is injective, \( A \) and \( \overline{A} \) may be identified).

2.12.b Let \( \mathfrak{X} = \mathbb{T}^{\mathbb{Z}} \) be as in 2.11.d and assume \( \varphi \) to be the shift. Furthermore \( \mathbb{H} := \{ \varphi \in \mathfrak{X} \mid \text{card} \{ k \mid \varphi(k) \neq e \} < \infty \} \). The exponential map \( \exp : \mathbb{R}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}} = \mathbb{R}^{\mathbb{Z}} / \mathbb{Z}^{\mathbb{Z}} \) is not injective (as in 2.12.a), but here also \( \exp : \mathfrak{g} = \mathbb{H} \) is surjective but not injective, where

\[ \mathfrak{g} := \{ \psi \in \mathbb{R}^{\mathbb{Z}} \mid \text{card} \{ k \mid \varphi(k) \neq e \} < \infty \} \]

is the Lie algebra corresponding to \( \mathbb{H} \). Let \( \varphi \) be the shift on \( \mathbb{R}^{\mathbb{Z}} \). Then \( \varphi \) acts contracting on \( \mathfrak{g} \).

Let \( B \in \mathcal{M}_0(\mathfrak{X}) \) be a generating distribution, such that

\[ \mu_t := \mathbb{E}(\exp(tB))_{t \geq 0} \]

is concentrated on a finite dimensional torus \( \mathbb{H}_0 := \mathbb{T}^{\mathbb{Z}}_1 \), \( I \subseteq \mathbb{Z} \). Assume \( c \in (0,1) \) and put \( A := \sum_{k \in \mathbb{Z}} c^{-k} \varphi^k(B) \).

(The sum is convergent as for any \( f \in \mathcal{A}(\mathfrak{X}) \) the number of non-zero summands is finite). Assume that \( \overline{B} \) is any generating distribution on (the finite dimensional vector space) \( \mathbb{R}^I \), such that \( \exp(\overline{B}) = B \). If we put

\[ \overline{A} := \sum_{k \in \mathbb{Z}} c^{-k} \varphi^k(\overline{B}) \text{, then exp}(\overline{A}) = A \text{, and obviously } \varphi(\overline{A}) = cA \text{, } \varphi(\overline{A}) = cA. \]

Assume e.g. \( \lambda \) to be a probability measure concentrated on \( (0,2\pi) \subseteq \mathbb{R} \) and put \( B := \lambda - e_0 \), \( B := \exp(\lambda - e_0) \in \mathcal{M}_0(\mathbb{T}) \). If we identify \( \mathbb{R} \) with \( \mathbb{R}^{\mathbb{Z}} \), \( \mathbb{T} \) with \( \mathbb{T}^{\mathbb{Z}} \), \( \mathcal{M}_0(\mathfrak{X}) \) with \( \mathcal{M}_0(\mathbb{R}^{\mathbb{Z}}) \), we get a semistable distribution \( A = \sum c^{-k} \varphi^k(B) \in \mathcal{M}_0(\mathbb{R}^{\mathbb{Z}}) \) and a semistable distribution \( A = \exp(\overline{A}) \in \mathcal{M}_0(\mathfrak{X}) \), such that no finite dimensional marginal distribution is semistable.
2.13. Remark Combining 2.10 and 2.11 we obtain the following result:

We start with $\mathcal{X} = \mathbb{T}^\mathbb{Z}$, and let $\varrho$ be the shift as before. Now we define for a non-countable set $I$ $G := \mathcal{X}^I$, and define $\tau$ as in 2.10. The $G$ is an infinite dimensional torus, therefore arcwise connected, but there exist semistable generating distributions $A$ w.r.t. $\tau$, such that $A$ is not concentrated on a measurable subset on which $\tau$ acts contracting.

§3 Limit theorems.

Now we want to indicate that there is a connection between limit theorems of distributions of normalized products and of semistable resp. selfdecomposable distributions. If $\mu \in M^1(\mathbb{R}^N)$ is (operator) self-decomposable then $\mu$ is representable as limit distribution of a sequence of normalized sums

$$Y_n = \sum_{k=1}^{n} A_n(X_k) + a_n,$$

where $(X_k)_{k=1}^{\infty}$ is a sequence of independent $\mathbb{R}^N$-valued random variables, $a_n \in \mathbb{R}^N$ and $A_n \in \text{Aut}(\mathbb{R}^N)$, such that $(A_n X_k, 1 \leq k \leq n)$ is uniformly infinitesimal. (See K. Urbanik [19].) $\mu$ is infinitely divisible, hence there exists a unique c.c.s. $(\mu_t) \in M^1(\mathbb{R}^N)$ such that $\mu_1 = \mu$. Moreover there exists a continuous multiplicative group $(\tau_t)_{t>0} \subseteq \text{Aut}(\mathbb{R}^N)$, such that

$$\mu_s = \tau_t(\mu_s) * \nu_{t,s}, \ s > 0, 0 < t < 1, \ \nu_{t,s} \in M^1(\mathbb{R}^N).$$

If we represent $(\mu_s)$ by the generating distribution $\mu_s = \mathcal{E}xp(sA)$, we obtain $A = \tau_t(A) + B(t)$ for some $B(t) \in M^0(\mathbb{R}^N)$, i.e. $A \in \mathcal{D}(\tau_t))$.

Now we give, following the ideas of [19], a simple generalization of this fact:

3.1. Theorem. Assume that $G$ is a locally compact group, $(\tau_t)_{t>0} \subseteq \text{Aut}(G)$, and assume further $A \in \mathcal{D}(\tau_t))$, i.e. $A = \tau_t(A) + B(t)$, $t \in (0,1)$. And assume finally that $\lim_{t \to 0} \tau_t(A) = 0$ in $\mathcal{C}(\mathcal{D}^\prime, \mathcal{D})$.

Then there exist $t_n > 0$, $t_n \to 0$,

such that (1). $A = \sum_{k=1}^{n} [\tau_{t_n}(C_k)]$ for some $C_k \in M^0(G)$,

$$\lim_{n \to \infty} \tau_{t_n}(C_k) = 0 \text{ in } \mathcal{C}(\mathcal{D}^\prime, \mathcal{D}), \ k \in \mathbb{N}.$$

Remark: If the generating distributions commute (especially if $G$ is abelian) the theorem becomes more familiar: Put

$$\nu(k) := \mathcal{E}xp(C_k), \ k = 1,2,\ldots.$$ Then $\mu_1 = \sum_{k=1}^{n} \nu_{t_n}(\nu(k)) = \tau_t(\sum_{k=1}^{n} \nu(k))$. 
Proof of the theorem:

\[ A = \tau_t(A) + B(t), \quad 0 < t < 1. \] Especially for \( t = e^{-1/m} \)

\[ A = \tau_t^{e^{-1/m}}(A) + B(e^{-1/m}) = \tau_t^{e^{-1/m}}(\tau_t^{e^{-1/m-1}}(A) + B(e^{-1/m-1})) + B(e^{-1/m}) = \]

\[ = \tau_t^{m}(A) + \tau_t^{m}(B(e^{-1})) + \cdots + \tau_t^{m}(B(e^{-1/m-1})) + B(e^{-1/m}) \]

\[ = \tau_t^{m}(C_1 + C_2 + \cdots + C_m + C_{m+1}), \quad e^{-s_m} \]

where \( s_m := \Sigma \frac{1}{k}, \quad C_1 = A, \quad C_2 = \tau_t^{e^{-1}}(B(e^{-1})), \ldots \)

\[ \ldots, \quad C_m = \tau_t^{e^{-1/m-1}}(B(e^{-1/m-1})), \quad C_{m+1} = \tau_t^{e^{-1/m}}(B(e^{-1/m})). \]

Put now \( t_{m+1} := e^{-s_m} \). Then we obtain \( t_{m+1}, \quad C_k \in \mathcal{M}_0(G), \quad 1 \leq k \leq m \)

and \( A = \tau_t^{m}(\Sigma_{k=1}^{m} C_k) \).

We have

\[ \tau_t^{m}(C_k) = \begin{cases} \tau_t^{m}(A) & k = 1 \\ \tau_t^{m} + \Sigma \frac{1}{k} (B(e^{-1/k-1})) & 1 < k \leq m \end{cases} \]

If \( k = 1, \quad f \in \mathcal{D}(G) \), we have \( < \tau_t^{m}(A), f > \rightarrow 0, \quad m \rightarrow \infty \).

If \( 1 < k \leq m \)

\[ A = \tau_t^{e^{-1/k-1}}(A) + B(e^{-1/k-1}), \quad \text{therefore} \]

\[ \text{(with } u_{k,m} := e^{-\Sigma \frac{1}{k}}, \quad u_{k,m} \rightarrow 0 \text{) } \tau_t^{m}(C_k) = \tau_t^{m}(B(e^{-1/k-1})) = \]

\[ \tau_t^{m}(A) - \tau_t^{m}(A) = \tau_t^{m}(A) - \tau_t^{m}(A), \quad \text{Hence we have for } f \in \mathcal{D}(G) : < \tau_t^{m}(C_k), f > \rightarrow 0. \]

Similar results are obtained for semistable distributions:

3.2 Theorem Let \( \tau \) be an automorphism of \( G \), \( \epsilon \in (0,1) \) and assume \( A \in \mathcal{M}_0(G) \) to be semistable w.r.t. \( (\tau, \epsilon) \). Assume that for \( Y \epsilon \mathcal{Y} \) \( A \cdot Y \) is concentrated on a set, on which \( \tau \) acts contracting, especially

\[ \tau^n(A \cdot Y) \rightarrow \epsilon \quad \text{in } \mathcal{E}(\mathcal{D}', \mathcal{D}). \]

Then there exists a sequence \( (k_n) \subseteq \mathbb{N}, \quad k_n \rightarrow c \), such that \( \frac{k_n}{k_{n+1}} \rightarrow c \),
and a sequence $Y_n \subseteq U_n$, such that
\[
A = \lim_{n \to \infty} k_n (\tau^n(A) + Y_n).
\]
If $A$ is strictly semistable, $Y = 0$, then we can choose $Y_n = 0$ and obtain
the more familiar form (see Jajte [9]): Put for fixed $t > 0$ $\mu = \mu_t = \mathcal{E}(tA)$
$\tau_n := \tau^n$. Then $\mu = \lim_{n \to \infty} [\tau_n(\mu)] k_n$.

Proof: We have for $n \in \mathbb{N}$: $\tau^n(A) = c^nA + X_n$, resp. $A = c^{-n}(\tau^n(A) - X_n)$.
Hence $A = c^{-n}\tau^n(A - Y) - c^{-n}X_n + c^{-n}\tau^n(Y)$.
Put $k_n := [c^{-n}]$, $n \in \mathbb{N}$ and $Y_n := \tau^n(Y) (c^{-n} / k_n - 1) - c^{-n} / k_n X_n$.
Then we obtain immediately $k_n(\tau^n(A) + Y_n) \to A$:
We have $k_n(\tau^n(A) + Y_n) = k_n\tau^n(A) + \tau^n(Y) (c^{-n} - k_n) - c^{-n} X_n =
= c^{-n}\tau^n(A) - c^{-n}X_n - \tau^n(A)(c^{-n} - k_n) + \tau^n(Y)(c^{-n} - k_n) =
= A - \tau^n(A - Y) \cdot (c^{-n} - [c^{-n}]) \to A$. □

As in the case of stable distributions (see [4]) an extension of
this result is possible:

3.3. Theorem. Let $G$ be a locally compact group,
c E (0,1), $\tau \in \text{Aut}(G)$ and $A \in \mathcal{M}_c(G)$.
Let $\{\nu(n)\} \subseteq \mathcal{M}^1(G)$ be a sequence of probabilities, and assume that
$\{k_n\} \subseteq \mathbb{N}$ is a sequence of integers, such that $k_n \to c$ and $k_{n+1}/k_n \to c$.
Assume further that for $t \in \mathbb{R}_+$
\[
\begin{aligned}
\{k_n t\} & \to \mathcal{E}(tA) \text{ in } \mathcal{E}(\mathcal{M}_c^b(G), \mathcal{C}_0(G)) \text{ uniformly on compact sets of } \mathbb{R}_+. \text{ Then } A \text{ is strictly semistable w.r.t. } (\tau,c).
\end{aligned}
\]

Proof: We know (see e.g. [3] 1.4.3) that the approximation of
\[
(\mathcal{E}(tA))_{t \geq 0} \text{ by "discrete semigroups" } (\tau^n(\nu))
\]
implies
\[
k_n(\tau^n(\nu) - \varepsilon_e) \text{ in } \mathcal{E}(\mathcal{D}', \mathcal{D}). \text{ Therefore } \tau(k_n(\tau^n(\nu) - \varepsilon_e)) \to \tau(A),
\]
hence
\[
\lim_{n \to \infty} \left(\frac{k_n}{k_{n+1}}\right) k_{n+1} (\tau^{n+1}(\nu) - \varepsilon_e) = \tau(A).
\]
Hence, as $k_n / k_{n+1} \to c$, $\tau(A) = cA$. □

Remarks:
1. The proof shows that the following holds:
If $A,B \in \mathcal{M}_c(G)$, $\tau \in \text{Aut}(G)$, $c \in (0,1)$ and $k_n \to c$ such that $k_n / k_{n+1} \to c$,
and if $k_n \tau^n(B) \to A$, then $\tau(A) = c \cdot A$

2. An extension of 3.3 is possible for not necessarily strictly semi-stable distributions. Under additional conditions the limits of centered discrete semigroups

$$\lim (\tau^n(\nu \ast \epsilon_{x_n}))^{[k_n t]} = \mathcal{E}\text{xp}(tA)$$

are semistable.

In the following theorem we indicate, how the results of §1 can be used to obtain limit theorems for random products with a random number of factors. We restrict to the case of semistable distributions.

From [6] 1.13, 1.24 and especially 5.16 we obtain:

3.4. Assume $s_n \to 0$, $t_n \to 0$, $v(n) \in \mathcal{M}^1(G)$, such that

$$\langle v(n) \rangle^{[s/s_n]} \mu_s \text{ weakly, uniformly on compact subsets of } \mathbb{R}^+,$$

where $(\mu_s)_{s \geq 0}$ is a continuous convolution semigroup in $\mathcal{M}^1(G)$. Assume $\delta(n) \in \mathcal{M}^1(\mathbb{R}^+)$, $(n_t)_{t \geq 0}$ to be c.c.s. in $\mathcal{M}^1(\mathbb{R}^+)$ such that $(\delta(n))^{[t/t_n]} \rightarrow \eta_t$. Then the mixtures converge:

$$\lim_{n, m \to \infty} \int_{\mathbb{R}^+} d\delta(m)^{[t/t_m]}(s) = \int_{\mathbb{R}^+} \mu_s \eta_t(s).$$

For special choice of $s_n, t_n, v(n), \delta(n)$ we obtain the following result:

3.5. Theorem. Assume $\tau \in \text{Aut}(G)$ and assume that $(\mu_s = \mathcal{E}\text{xp}(sA)) \leq \mathcal{M}^1(G)$ is strictly semistable w.r.t. $(\tau, c)$. Assume further that $v \in \mathcal{M}^1(G)$ is such that for $k_n \to \infty$, $\tau^n(\nu)^{[k_n s]} \rightarrow \mu_s$.

Let $(n_t)_{t \geq 0} \leq \mathcal{M}^1(\mathbb{R}^+)$ be a semistable c.c.s. w.r.t. $(\delta_u, d)$, $(\delta_u$ denotes the automorphism $x \mapsto u x$, for fixed $u > 0$), and $1 > d > 0$.

Assume that $\delta \in \mathcal{M}^1(\mathbb{R}^+)$ such that

$$\langle \delta_{u_n(\delta)} \rangle^{[1]} \rightarrow \mu_1 \text{ (and hence } \langle \delta_{u_n(\delta)} \rangle^{[1/n t]} \rightarrow \eta_t)$$

for a sequence $l_n \rightarrow \infty$, $l_n / l_{n+1} \rightarrow d$.

Then the random products

$$\int_{\mathbb{R}^+} \tau^n(\nu)^{[k_n s]} \cdot d(\delta^{[1/n t]}(s/y_m))$$

converge to the subordinated semigroup...
Prop. 1.11 shows that \((\lambda_t)\) is a semistable convolution semigroup.

The proof follows immediately from 3.4:

We put \(\nu^{(n)} := \tau^n(\nu)\), \(\mathfrak{c}^{(n)} := \delta_{u^n(\mathfrak{c})}\), \(s_n := 1/k_n\), \(t_n := 1/l_n\).

3.6. Corollary. Assume that \(G\) is metrizable. Assume further \(\tau, \nu, \delta_u, \mathfrak{c}, (\pi_t), (\mu_t)\) to be given as before.

Assume \((\Omega, \Sigma, P)\) to be a probability space, \((X_i)_{i=0}^\infty, (Y_j)_{j=0}^\infty\) to be a set of independent random variables, \(X_i : \Omega \to G\), \(Y_i : \Omega \to \mathbb{R}_+\) such that \(X_0 \equiv e\), \(Y_0 \equiv 0\), \(X_i(P) = \nu, i \in \mathbb{N}\), \(Y_i(P) = \mathfrak{c}, i \in \mathbb{N}\).

Define a sequence of \(G\) - valued stochastic processes via

\[X(n)(s, \omega) := \tau^n(X_0(\omega) \ldots X_{\lfloor k_n s \rfloor}(\omega)),\]

and an sequence of \(\mathbb{R}_+\) - valued processes via

\[Y(n)(t, \omega) := u^n(\sum_{0}^{\lfloor l_n t \rfloor} Y_i).\]

Finally the sequence of subordinated processes is defined via

\[Z(n,m)(t, \omega) := X(n)(Y(m)(t, \omega), \omega) = \tau^n(X_0(\omega) \ldots X_{\lfloor k_n Y(m)(t, \omega) \rfloor}(\omega)).\]

Let \(X(s, \cdot)\) be a \(G\) - valued process with independent increments according to \((\mu_t)\) and let \(Y(t, \cdot)\) be an independent \(\mathbb{R}_+\) - valued process with independent increments according to \((\pi_t)\), and define the subordinated process \(Z(t, \omega) := X(Y(t, \omega), \omega)\).

Then we obtain from 3.4, that the normalized random products \(Z(n,m)(t, \omega)\) converge in distribution to the subordinated process \(Z(t, \omega)\).
Literature

0. CH. BERG, G. FORST: "Multiply self-decomposable probability measures on $\mathbb{R}_+$ and $\mathbb{Z}_+$."


Let \(( B, \| \| \) \) be a real separable Banach space equipped with its Borel \( \sigma \)-field \( \mathcal{B} \) and consider \(( X_n, n \in \mathbb{N} )\) a sequence of independent \(( B, \mathcal{B} )\) - valued random variables \((r.v.)\) defined on a probability space \(( \Omega, \mathcal{F}, P )\). For every integer \( n \) we define :

\[ S_n = X_1 + X_2 + \ldots + X_n. \]

The sequence \(( X_n, n \in \mathbb{N} )\) satisfies the weak law of large numbers \(( (X_n) \in \text{WLLN} )\) if and only if :

\[ \frac{S_n}{n} \overset{p}{\to} 0; \]

it satisfies the strong law of large numbers \(( (X_n) \in \text{SLLN} )\) if and only if :

\[ \frac{S_n}{n} \overset{a.s.}{\to} 0. \]

The aim of the present paper is to study the SLLN in 2-uniformly smooth Banach spaces. We will first recall some of the classical results on the SLLN for Banach space valued r.v. and we will also give some examples which cannot be reached by them. This introductory section will justify our interest in r.v. with values in 2-uniformly smooth Banach spaces, for which the results known previously in general Banach spaces can be considerably improved.

Let's begin this short survey ( a more complete one can be found in [27] )
by some results which don't assume any restriction on the space $B$.

RESULT 1: The i.i.d. situation: E. Mourier's Theorem [22]:

For every sequence $(X_n)$ of independent copies of a $B$-valued r.v. $X$ one has:

$$E SLLN \iff E \|X\| < \infty \text{ and } E(X) = 0.$$ 

RESULT 2: The non i.i.d. situation: J. Kuelbs and J. Zinn's [18] extension of Prohorov's Theorem [24]:

Let $(X_n)$ be a sequence of independent $(B, \|\|)$ valued r.v. such that:

$$\exists M < \infty: \forall j \in \mathbb{N} \quad \|X_j\| \leq M (\log^2 j) \text{ a.s.}$$

(Where $\log x = \log (\sup(e, \log x))$).

Furthermore we suppose that for every $\epsilon > 0$ one has:

$$\sum_{n \geq 1} \exp - (\epsilon / \Lambda(n)) < \infty,$$

with:

$$\Lambda(n) = 2^{-2(n+1)} \sum_{j \in I(n)} E \|X_j\|^2,$$

and:

$$I(n) = \{ 2^{n+1}, \ldots, 2^{n+1} \}.$$ 

Then:

$$(X_n) \in SLLN \iff (X_n) \in WLLN.$$ 

Now we turn our attention to results which need restrictions on the space $(B, \|\|)$.

The first kind of restriction we will consider on $(B, \|\|)$ is $B$-convexity. Remember that $(B, \|\|)$ is $B$-convex if there exists an integer $k$ and $\epsilon \in \]0,1[$ such that:

$$\forall a_1, \ldots, a_k \in B: \sup \|a_i\| \leq 1, \quad \inf_{i} \epsilon_i a_i \| \leq k (1- \epsilon)$$
In this setting we will recall 2 theorems which are due to A. Beck [4] and W. A. Woyczynski [26] respectively.

RESULT 3 : ( A. Beck ) : The following are equivalent for a Banach space ( B, || || ):
1) B is B-convex .
2) Every sequence (Xₙ) of independent, centered B-valued r.v. such that
\[ \sup_n B \|X_n\|^2 < + \infty , \] satisfies the SLLN.

RESULT 4 : ( W. A. Woyczynski ) : The following are equivalent for a Banach space
( B, || || ) :
1) B is B-convex .
2) If (Xₙ) is any sequence of independent, centered B-valued r.v. such that there exists a positive r.v. X₀ with :
i) \[ E X₀ \log^+ X₀ < + \infty ; \]
ii) \[ \exists C > 0 : \forall t > 0 , \forall i \quad P( \|X_i\| > t ) \leq C P( X₀ > t ) ; \]
then \( (Xₙ) \in \text{SLLN} \).

What happens now if we make another kind of assumption on \( (B, || ||) \) : \( l^p \) is not finitely representable in \( (B, || ||) \) ?

Let \( p \in [1,2] \); one says that \( l^p \) is finitely representable in \( B \) if :
\[ \forall E > 0 , \forall n \in \mathbb{N} , \exists x₁, \ldots, xₙ \in B : \]
\[ \forall α₁, \ldots, αₙ \in \mathbb{R} : ( \sum_{1 \leq i \leq n} |α_i|^p )^{1/p} \leq \| \sum_{1 \leq i \leq n} α_i x_i \| \leq (1+ε)( \sum_{1 \leq i \leq n} |α_i|^p )^{1/p} \]

This property and the B-convexity are closely related. More precisely, \( (B, || ||) \) is B-convex if and only if \( l^1 \) is not finitely representable in \( B \).

For this class of Banach spaces the following results hold :
RESULT 5: (B. Maurey, G. Pisier [20]): Let $1 \leq p < 2$. The following are equivalent for a Banach space $(B, \| \cdot \|)$:

1) $l^p$ is not finitely representable in $B$.

2) $n^{-1/p} \left( \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n \right) \xrightarrow{a.s.} 0$ for any sequence of independent Rademacher r.v. $(\varepsilon_n)$ and bounded sequence $(x_n)$ of elements belonging to $B$.

RESULT 6: (W. A. Woyczynski [26]): Let $1 < p < 2$. The following are equivalent for a Banach space $(B, \| \cdot \|)$:

1) $l^p$ is not finitely representable in $B$.

2) If $(X_n)$ is any sequence of independent, centered $B$-valued r.v. such that there exists a positive r.v. $X_0 \in l^p$ with:
$$\forall i, \forall t > 0 \quad P(\|X_i\| > t) \leq C P(X_0 > t),$$
then:
$$n^{-1/p} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} 0.$$

The last class of special Banach spaces that we will consider is probably the most famous one (at least for Banach probabilists!): the type $p$ spaces. Let $1 \leq p \leq 2$. The space $(B, \| \cdot \|)$ is of type $p$ if there exists $C > 0$ such that for every finite sequence $(X_1, \ldots, X_n)$ of independent centered $B$-valued r.v.:
$$E \left\| \sum_{k=1}^{n} X_k \right\|^p \leq C \sum_{k=1}^{n} E \|X_k\|^p.$$

Notice that $B$ is $B$-convex if and only if there exists $p > 1$ such that $B$ is of type $p$.

In these spaces also we will recall two famous results.

RESULT 7: (J. Hoffmann-Jørgensen, J. Pisier [13]): Let $1 \leq p \leq 2$. The following are equivalent for a Banach space $(B, \| \cdot \|)$:
RESULT 8: The extension of the Marcinkiewicz – Zygmund SLLN to type \( p \) spaces 
\((1 < p < 2)\).

Many people have worked on this problem and have obtained partial results (see [3] for a short history); A. de Acosta [1] and T. A. Azlarov and N.A. Volodin [3] made the major contributions to its solution.

The final result [1] is as follows:

Let \( 1 < p < 2 \). The following are equivalent for a Banach space \((B, \|\|)\):

1) \( B \) is of type \( p \).

2) Every sequence \((X_n)\) of i.i.d., centered, \( B \)-valued r.v. with \( \mathbb{E} \|X_1\|^p < \infty \) is such that: \( n^{-1/p} S_n \xrightarrow{a.s.} 0 \).

The conclusion of this short survey of results is that the situation is relatively satisfactory in the i.i.d. or "asymptotically" i.i.d. cases. We will now give an example showing that in the non-i.i.d. case the results are not as good.

It will be an example of a sequence in \( c_0 \), which satisfies the SLLN, but which doesn't fulfil the hypothesis of the result 2 above [18].

Consider \((e_n)\) the canonical basis of \( c_0 \) and \((\xi_n)\) a sequence of independent Rademacher r.v.. Define:

\[ X_n = \left( \frac{n}{L_2^n} \right) \xi_n e_n \]

It is clear that:

\[ \forall j, \quad \|X_j\| \leq j/L_2^j \]
but: \[ A(n) = 2^{-2(n+1)} \sum_{j \in I(n)} (j^2/(L_2 j)^2) \geq C \left( 2^{n/(\log n)^2} \right); \]

and therefore: \[ \forall \varepsilon > 0, \sum_{n=1}^{\infty} \exp(-\varepsilon / A(n)) = +\infty. \]

On the other hand:

\[ \forall n, \quad \|S_n / n\| = 1/ L_2 n, \]

and obviously \( (X_n) \in SLLN. \)

One could object that the space we have chosen for constructing this example is the worst possible. So let's give another example showing that the situation is not better in a "more reasonable space": \( l^2 \) equipped with its usual basis \( (e_n). \)

Let \( (\lambda^j_n, j = 1, \ldots, [n^{1/6}] - [ ] \) denoting the integer part of a number - be a triangular array of real valued r.v., the lines being independent, and the r.v. in each line being also independent. All the r.v. of the n-th line have the same distribution, that is the one of \( X I_{(|X| \leq (n^{5/6} / L_3 n)} \), where \( X \) is a Cauchy r.v. and \( L_3 \) denotes the function \( \log L_2 \).

Now we define our sequence of \( l^2 \) -valued r.v.:

\[ X_n = \sum_{1 \leq j \leq [n^{1/6}]} \lambda^j_n e_j. \]

It is easy to see that:

i) \[ \exists M > 0 : \forall n \quad \|X_n\| \leq M \left( n / L_2 n \right), \]

ii) \[ \exists M' > 0 : \forall n \quad E \|X_n\|^2 \geq M' \left( n / L_3 n \right). \]

It follows that there exists an \( \varepsilon > 0 \) such that the series with general term \( \exp(-\varepsilon / A(n)) \) diverges. We will see later that \( (X_n) \in SLLN \); so here again result 2 doesn't allow to conclude. The space \( l^2 \) being of type 2, one can try to check if result 7 applies to this example; it is easy to see that it doesn't.

For reaching such situations we need results which don't integrate into a
"few geometry of the space" only - for example by applying roughly the definition of the type - but which take into account hypotheses on the finite dimensional projections of the r.v.

In order to develop this idea one can look at what has been done in this direction for the 2 other famous limit theorems: the central-limit theorem and the law of the iterated logarithm. The work done by J. Kuelbs [14], [15], J. Hoffmann-Jörgensen [12], V. Goodman, J. Kuelbs and J. Zinn [9], A. de Acosta and J. Kuelbs [2], and M. Ledoux [19] shows that the central-limit theorem and the law of the iterated logarithm can be studied under good hypotheses in spaces whose norm has good Frechet derivatives.

For the central-limit theorem and the law of the iterated logarithm, the most handy class of such spaces with a regular norm seems to be the class of 2-uniformly smooth spaces.

Our goal is to show that the SLLN also can be studied nicely in 2-uniformly smooth Banach spaces.

Before to state the results we will recall shortly some properties of the 2-uniformly smooth spaces. The reader will find more details in [19], [15] and of course in the pioneer work of R. Fortet and E. Mourier [8].

**DEFINITION**: A Banach space \((B, \| \cdot \|)\) of dimension \(n \geq 2\) is 2-uniformly smooth if there exists a positive constant \(K\) such that:

\[
\forall (x,y) \in B^2 \quad \|x+y\|^2 + \|x-y\|^2 \leq 2 \|x\|^2 + K \|y\|^2.
\]

These spaces have a lot of interesting properties; we recall the ones we will need later:

**PROPERTY 1**: There exists \(D : B - \{0\} \to B'\) such that:
\forall (x,y) \in B^2, \forall t \in \mathbb{R} : x + ty \neq 0, \frac{d}{dt} \| x + ty \| = D(x+ty)(y) .

PROPERTY 2 : If one defines \( F : B \to B' \) by :

\[ F(x) = \| x \| D(x/\| x \|), \quad F(0) = 0, \]

then :

i) \( F(x)(x) = \| x \|^2 \),

ii) \( \| F(x) \|_B = \| x \| \),

iii) \( \exists C > 0 : \forall (x,y) \in B^2 \| F(x) - F(y) \|_B \leq C \| x - y \| . \)

For suitability we will call every \( C \) fulfilling this inequality a smoothness constant for the space \((B, \| \|)\).

PROPERTY 3 : A 2-uniformly smooth space is of type 2. If \( C \) is the constant involved in the type inequality, \( C \) is also a smoothness constant for the space.

PROPERTY 4 : The fundamental inequality [19]. For every finite sequence \((x_1, \ldots, x_n)\) of elements in \( B \), one has :

\[ \left\| \sum_{1 \leq j \leq n} x_j \right\|^2 \leq 2 \sum_{1 \leq j \leq n} F(\sum_{1 \leq k \leq j-1} x_k)(x_j) + C \sum_{1 \leq j \leq n} \left\| x_j \right\|^2 , \]

where : \( \sum_{1 \leq k \leq 0} x_k = 0 \).

Now we can begin to state our results.

1. THE KOLMOGOROV SLLN IN 2-UNIFORMLY SMOOTH BANACH SPACES.

In a 2-uniformly smooth Banach space, the well known Kolmogorov SLLN can be stated in the following way:
THEOREM 1: Let \((X_n)\) be a sequence of independent, centered r.v. with values in a 2-uniformly smooth Banach space \((B, \|\|)\). Suppose that the following hold:

a) \(\exists K > 0 : \forall n \in \mathbb{N} \quad \|X_n\| \leq K \left( \frac{n}{(\log n)^{1/2}} \right) \quad a.s.\),

b) \(n^{-2} \sum_{1 \leq k \leq n} \|X_k\|^2 \to 0 \),

c) \(\sum_{j \geq 1} \sup \left( \mathbb{E}(|X_j|/j^2), \|X_j\|_B \leq 1 \right) < +\infty \).

Then \((X_n) \in \text{SLLN}\).

Remark 1: In [10] the preceding result is stated with a condition which is apparently more restrictive than b), that is:

b') \(n^{-2} \sum_{1 \leq k \leq n} \mathbb{E}\|X_k\|^2 \to 0 \).

The first step of the proof of Theorem 1 will show that in fact a) and b) imply b').

Proof of Theorem 1: We begin with:

LEMMA 1: For every \(n\) we put:

\[ Z_n = n^{-2} \sum_{1 \leq k \leq n} \|X_k\|^2 \].

Then:

\[ \lim_{n \to \infty} \mathbb{E}(Z_n) = 0 \].

Proof of Lemma 1: Let \((X'_k)\) be an independent copy of the sequence \((X_k)\) and let \((Z'_n)\) be the sequence:

\[ Z'_n = n^{-2} \sum_{1 \leq k \leq n} \|X'_k\|^2 \].

One has obviously:

\[ (Z_n - Z'_n) \to 0 \].

By a well known result of A. de Acosta ([1] Lemma 3.1) it follows:

\[ \delta_n = \mathbb{E}(Z_n - Z'_n)^2 \to 0 \quad n \to \infty \].
If one denotes by \( \mu_n \) a median of the r.v. \( Z_n \) one has:
\[
\lim_{n \to \infty} \mu_n = 0,
\]
and by a classical symmetrization argument one obtains for every \( t > 0 \):
\[
P( Z_n - \mu_n > t ) \leq 4 P( |Z_n - Z_n^*| > t ) \leq 4 \delta_n / t^2.
\]
It follows from this inequality:
\[
E( Z_n ) \leq \sup \{ 2\mu_n, \delta_n \} + 4 \delta_n \delta_n\frac{1}{n},
\]
and this ends the proof of Lemma 1.

The space \( B \) being of type 2, it follows that \((X_n) \in WLLN\). So by a classical property ([18] Lemma 2.1) we can limit the proof of Theorem 1 to a sequence \((X_n)\) of symmetrically distributed r.v.'s.

Notice that by the fundamental inequality:
\[
\left\| \frac{S_n}{n} \right\|^2 \leq 2 \frac{1}{n} \sum_{1 \leq j \leq n} P(s_{j-1})(X_j) + C Z_n.
\]
We will study separately the two right-hand side terms.

**LEMMA 2**
\[
Z_n \overset{a.s.}{\to} 0.
\]

In fact we will show a stronger property:
\[
n^{-1} \sum_{1 \leq j \leq n} (\|X_j\|^2 / j) \overset{a.s.}{\to} 0.
\]

One first notices:
\[
\forall j \in N, \quad \|X_j\|^2 / j \leq \chi^2 (j / L_2 j) \text{ a.s. };
\]
and:
\[
\Lambda(n) = z^{-2(n+1)} \sum_{j \in I(n)} \mathbb{E} \left( \|X_j\|^2 / j \right)^2 \leq (k'/\log n) z^{-2(n+1)} \sum_{j \in I(n)} \mathbb{E} \|X_j\|^2.
\]
It follows:

\[ \forall \varepsilon > 0 \quad \sum_{n \geq 1} \exp - \left( \frac{\varepsilon}{A(n)} \right) < + \infty, \]

and the required convergence is an easy consequence of Result 2.

**Lemma 3:** \[ n^{-2} \sum_{1 \leq j \leq n} P(S_{j-1})(X_j) \xrightarrow{a.s.} 0. \]

This Lemma, whose proof follows from the martingale convergence theorem in [6], ends the demonstration of Theorem 1.

**Remark 2:** The \( l^2 \)-valued r.v. that we constructed earlier fulfill the assumptions of Theorem 1:

i) \[ \|X_n\| \leq K \left( \frac{n}{L_2 n^2} \right) \text{ a.s.,} \]

ii) \[ E \|X_n\|^2 = \sum_{1 \leq j \leq \lfloor n/6 \rfloor} E (\lambda_j^n)^2 \leq K \left( \frac{n}{L_3 n} \right), \]

and so: \[ \lim_{n \to \infty} E(Z_n) = 0. \]

iii) Now we have to bound the quantity \( \sup \left( E(f^2(X_n)) , \|f\|_{l^2} \leq 1 \right) \) which can be written in the following more detailed way:

\[ \sup \left( E\left( \sum_{1 \leq k \leq \lfloor n/6 \rfloor} a_k^2 (\lambda_k^n)^2 \right) , \text{ where } \sum_{1 \leq k \leq \lfloor n/6 \rfloor} a_k^2 \leq 1 \right). \]

This quantity is dominated by \( (2n^{5/6} / nL_3 n) \); therefore:

\[ \sum_{n \geq 1} \sup \left( \left( E(f^2(X_n)) / n^2 \right) , \|f\|_{l^2} \leq 1 \right) < + \infty. \]

All the hypotheses of Theorem 1 being fulfilled, \( (X_n) \in \text{SLLN}. \)

In the next section we will prove an exponential inequality which has its own interest, and which will be the crucial tool in the proof of the Prohorov SLLN in 2-uniformly smooth Banach spaces. In the appendix to this paper we will
also use this inequality for giving a very short proof of M. Ledoux's [19] law of the iterated logarithm in 2-uniformly smooth Banach spaces.

2. AN EXPONENTIAL INEQUALITY IN 2-UNIFORMLY SMOOTH BANACH SPACES.

Following the work of V.V. Yurinskii [28], there were a lot of papers which deal with extensions to Banach space valued r.v. of the well known Bernstein inequality for sums of real valued r.v.. These results apply to any Banach space and involve only the norm of the r.v.. Here we will state an inequality which involves weak-square integrability properties of the r.v., but which applies only to 2-uniformly smooth Banach spaces. Of course this inequality is much stronger than the general results known previously.

THEOREM 2: Let C, K, n be 3 positive integers fulfilling:

i) \( 144 \leq C < \left( \frac{L_2 n}{32} \right) \),

ii) \( 16 \leq K \leq \left( \left( \frac{L_2 n}{32} \right)^{\frac{3}{2}} / 128 \right) \).

Let now \(( B, \| \| )\) be a 2-uniformly smooth Banach space admitting C as a smoothness constant. Consider \( X_1, \ldots, X_n \) independent, symmetrically distributed \( B\)-valued r.v. such that:

\[
\forall j = 1, \ldots, n \quad \|X_j\| \leq K \left( \frac{a}{L_2 n} \right) \text{ a.s.},
\]

where \( a \) is a positive constant.

Define the quantities:

\[
\gamma = K \exp \left( \frac{K}{2} \right),
\]

\[
\Lambda = a^{-2} \sum_{1 \leq j \leq n} \sup_{\|f\|_B \leq 1} \mathbb{E} f^2(X_j),
\]

\[
\lambda = a^{-2} \sum_{1 \leq j \leq n} \mathbb{E} \|X_j\|^2.
\]
If $\Lambda \leq e^{-C}$ and $\Lambda \leq \frac{1}{2}$, the following hold:

1) $(K / L_2 n) \leq \gamma A \Rightarrow P(\sum_{1 \leq k \leq n} F(S_{k-1})(X_k) > 16a^2 C) \leq 4(\log C (1/ A)) \exp - (4/ \gamma A)$,

where $\log C$ denotes the logarithm with basis $C$ and $S_0 = 0$.

2) $(K / L_2 n) > \gamma A \Rightarrow P(\sum_{1 \leq k \leq n} F(S_{k-1})(X_k) > 16a^2 C) \leq 4(\log n)^{-3/2}$.

Remarks 3:
- The assumption $C \geq 144$ is not a loss of generality on the space $(B, \| \| )$, because if $C$ is a smoothness constant for $B$, $C' > C$ of course is one also.
- The assumptions $\Lambda \leq e^{-C}$ and $\Lambda \leq \frac{1}{2}$ aren't restrictive in the scope in which we will use Theorem 2, that is for $\Lambda$ and $\Lambda$ very close to 0.

In spite of the fact that Theorem 2 is a natural extension of the ideas we developed in [11], we will give the main steps of its proof for sake of completeness.

Proof of Theorem 2: Denote by $r$ the quantity:

$$r = P(\sum_{1 \leq k \leq n} F(S_{k-1})(X_k) > 16a^2 C).$$

We begin by the case: $(K / L_2 n) \leq \gamma A$.

We will define a real valued martingale (with respect to the increasing family of $\sigma$-fields $\mathcal{F}_j = \sigma(X_1, \ldots, X_j)$) in the following way:

$$\xi_0 = 0,$$

$$\xi_k = a^{-2} \sum_{1 \leq j \leq k} F(S_{j-1})(X_j) I(\|S_{j-1}\| \leq (aL_2 n / k)), $$

$$\forall k > n \quad \xi_k = \xi_n.$$

By symmetry, an immediate application of V.V. Yurinskii's exponential inequality [28] gives:
Now we want to bound $P(\xi_n \geq 16C)$ by using a martingale technique. For this we need some complementary notations.

We will denote by $\varphi$ the function: $R \to R$

$$t \to \exp(t - 1 - t);$$

and by $\mu$ a positive number which will be specified later; furthermore $\mu' = 2\mu$.

For every $j$, $2 \leq j \leq n$, the following inequality holds a.s.:

$$g_j = 1 + \mathbb{E}(\varphi(\mu/a^2)P(\xi_{j-1})(X_j)I(\|s_{j-1}\| \leq (aL_n)/x)) | \mathcal{F}_{j-1})$$

$$\leq 1 + \sum_{k=1}^{\infty} \left( \frac{\mu^2}{(2k)!}(\gamma A)^{2k-2} \left( \sup_{1 \leq k \leq n}(e_x/2a^2) + \sum_{1 \leq k \leq n}(\|X_k\|^2/4Ca^2) \right) \right) \sup_{1 \leq j \leq k}(\mathbb{E}[g_j^2(X_j)/a^2])$$

We will bound this latter quantity by splitting $\Omega$ into 2 sets, $\Omega_1$ and its complement:

$$\Omega_1 = \left( \sup_{1 \leq k \leq n}(e_x/2a^2) + \sum_{1 \leq k \leq n}(\|X_k\|^2/4Ca^2) \leq 1 \right), \text{with } \xi_k = a^{-2} \sum_{1 \leq j \leq k}P(\xi_{j-1})(X_j)$$

By choosing $\mu' = 1/\gamma A$, we obtain for every $w \in \Omega_1$:

$$\prod_{1 \leq j \leq n} g_j(w) \leq \exp(e/\gamma^2 A).$$

By applying now a martingale exponential inequality due to P.A. Meyer (Theorem 68), to the r.v. $(\exp(-\xi_n)/\prod_{1 \leq j \leq n} g_j)$, we finally obtain:

$$r \leq 2\exp(-(L_n)^2/16\cdot K^2) + P(\Omega_1^c) + \exp(-4/\gamma A),$$

and also:

$$r \leq 2\exp(-4/\gamma A) + P(\Omega_1^c).$$

In the next step of this proof we will dominate the quantity $P(\Omega_1^c)$.

Again by V.V. Yurinskii's exponential inequality:

$$P(\Omega_1^c) \leq P\left( \sum_{1 \leq k \leq n}(\|X_k\|^2 > 2Ca^2) \right) + P\left( \sup_{1 \leq k \leq n}(e_x^2(X_k)/a^2) \right)$$

$$\leq \exp(-(L_n)^2/16\cdot K^2) + P\left( \sup_{1 \leq k \leq n}(e_x^2(X_k)/a^2) \right)$$
\[ r \leq \exp(-4/\gamma A) + \Pr(\sup_{1 \leq k \leq n} \xi_k' > C^2). \]

and so:

1. \[ r \leq 3 \exp(-4/\gamma A) + \Pr(\sup_{1 \leq k \leq n} \xi_k' > C^2). \]

We notice that it remains to bound a term, \( \Pr(\sup_{1 \leq k \leq n} \xi_k' > C^2) \), which looks like \( r \) itself, the main difference being that \( C^2 \) is bigger than \( 16C \). The idea of the proof is to proceed by induction; at the first step we will treat the term \( \Pr(\sup_{1 \leq k \leq n} \xi_k' > C^2) \) by the same technique as \( r \) in order to obtain a formula similar to (1), but involving the quantity \( \Pr(\sup_{1 \leq k \leq n} \xi_k' > C^2) \). At the \( j \)-th step we will introduce the quantity \( \Pr(\sup_{1 \leq k \leq n} \xi_k' > C^{j+1}) \). The key-idea is again to split \( \Omega \) into 2 sets, a good one \( \Omega_j \) and a bad one \( \Omega_j^c \). More precisely:

\[ \Omega_{j+1} = \left( \sup_{1 \leq k \leq n} (\xi_k' / 2C^{j+2}) + \sum_{1 \leq k \leq n} (\|X_k\|^2 / 4C^{j+1}a^2) \leq 1 \right). \]

By the same martingale argument as before — with \( \mu = (1/2C^{j+2}a\gamma A) \) — we obtain that for every integer \( j \):

2. \[ r \leq 3j \exp(-4/\gamma A) + \Pr(\sup_{1 \leq k \leq n} \xi_k' > C^{j+1}). \]

By choosing now \( j = [\log(C(1/\Lambda))] \), we get:

\[ r \leq (3\log(C(1/\Lambda))) \exp(-4/\gamma A) + \Pr(\sup_{1 \leq k \leq n} \xi_k' > 1/\Lambda). \]

For bounding the last term in this relation we use another result of P.A. Meyer ([21] Theorem 69) and we obtain finally:

\[ r \leq 3(\log(C(1/\Lambda)) + 1) \exp(-4/\gamma A), \]

and this ends the proof of the first part of Theorem 2.

Part 2) of Theorem 2 is proved in a similar manner. The main difference is that the constants \( \mu \) have to be chosen differently: for instance at the first step in the induction we will take \( \mu' = \frac{1}{2} \mu n \cdot \). The relation (2) becomes:
Here we chose \( j = \left[ \frac{8L_2 n}{\log C} \right] + 1 \), and we obtain:

\[
 r \leq (\log n)^{-3/2} + P \left( \sup_{1 \leq k \leq n} \xi_k \geq (L_2 n)^8 \right). 
\]

Finally:

\[
 r \leq 4 (\log n)^{-3/2}. 
\]

Now we can state and prove Prohorov's SLLN in 2-uniformly smooth Banach spaces.

3. THE PROHOROV SLLN IN 2-UNIFORMLY SMOOTH BANACH SPACES.

Prohorov's SLLN extends to 2-uniformly smooth Banach spaces in the following way:

**THEOREM 3**: Let \((X_n)\) be a sequence of centered, independent r.v. with values in a 2-uniformly smooth Banach space \((B, \| \cdot \|)\). Suppose that the following hold:

i) \( \exists M > 0 : \forall k \in \mathbb{N}, \|X_k\| \leq (Mk / L_2 k) \) a.s.,

ii) \( n^{-2} \sum_{1 \leq k \leq n} \|X_k\|^2 \rightarrow 0 \),

iii) Define for every integer \( n \):

\[
\Lambda(n) = 2^{-2n} \sum_{j \in I(n)} \sup_{\|f\|_B \leq 1} \|F_j(x_j)\|
\]

then:

\[
\forall \varepsilon > 0, \sum_{n \geq 1} \exp(-\varepsilon / \Lambda(n)) < +\infty.
\]

Under these hypotheses \((X_n) \in \text{SLLN}\).
The proof is an easy application of Theorem 2.

As we have done in the proof of Theorem 1, we can again suppose that the $X_k$ are symmetrically distributed; furthermore, we need only to check:

$$\forall \varepsilon > 0 \quad \sum_{n=1}^{\infty} p\left( 2^{-n} \left\| \sum_{j \in I(n)} X_j \right\| > \varepsilon \right) < +\infty.$$ 

For bounding the general term of this series we define a sequence of auxiliary r.v.:

$$Y_k = \left( 2^{5/2} \varepsilon^{3/2} X_k / 2^n \right).$$

Then:

$$p\left( 2^{-n} \left\| \sum_{j \in I(n)} X_j \right\| > \varepsilon \right) \leq p\left( 2^{-2n} \sum_{j \in I(n)} \left\| X_j \right\|^2 > \varepsilon^2/2C \right) + p\left( \sum_{j \in I(n)} F(S_{j-1}(Y))(Y_j) > 16C \right).$$

From the proof of Theorem 1 it follows that:

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} p\left( 2^{-2n} \sum_{j \in I(n)} \left\| X_j \right\|^2 > \varepsilon^2/2C \right) < +\infty.$$ 

For $n$ large enough the family of r.v. $(Y_j, j \in I(n))$ fulfills all the assumptions of Theorem 2. So:

$$p\left( \sum_{j \in I(n)} F(S_{j-1}(Y))(Y_j) > 16C \right) \leq 4 \sup \left( \left( \log_2 \Lambda'(n) \right) \exp\left(-4/\varepsilon \Lambda'(n) \right), (n \log 2)^{-3/2} \right).$$

where:

$$\Lambda'(n) = 32C \varepsilon^{-2} \Lambda(n).$$

One has: $\lim_{n \to \infty} \Lambda(n) = 0$; so for $n$ large enough:

$$p\left( \sum_{j \in I(n)} F(S_{j-1}(Y))(Y_j) > 16C \right) \leq 4 \sup \left( \exp\left(-2\varepsilon^2/32C \Lambda'(n) \right), (n \log 2)^{-3/2} \right).$$

This is the general term of a convergent series; so the proof of Theorem 3 is completed.

A natural question raised by the statement of Theorem 3 is: "To what Banach spaces does it extend?"

A (very) partial answer to this question will be given by an example of
a $c_0$-valued r.v. that we will study now.

Suppose that for every integer $n$ a sequence of independent Rademacher r.v. $(\xi^n_j, j \in \mathbb{N})$ is given; furthermore all the sequences $(\xi^n_j)$ are supposed to be independent. The sets $I(n)$ being defined as before we put for every $k \in I(n)$:

$$X_k = \left( 2^n / (\log n) (L_2 n) \right)^{\frac{1}{2}} (\xi^1_k, \ldots, \xi^\alpha(n)_k, 0, 0, \ldots),$$

where $\alpha(n)$ denotes the integer part of $n^{L_2 n - 1}$ (and by convention $\log 0 = L_2 0 = 1$).

Finally:

$$X_1 = (0, \ldots, 0, \ldots).$$

First we check that this sequence $(X_n)$ of independent $c_0$-valued r.v. satisfies the conditions of Theorem 3.

It is clear that the $X_k$ are centered and that:

$$\exists \ M > 0 : \forall k \in \mathbb{N}, \|X_k\| \leq M (k^{L_2} k).$$

Let $k$ be a given integer which is larger or equal to 2. So it belongs to one of the $I(j), I(n)$ say. Then:

$$k^{-2} \sum_{1 \leq j \leq k} \mathbb{E}\|X_j\|^2 \leq 2^{-2n} \sum_{1 \leq j \leq 2n+1} \mathbb{E}\|X_j\|^2 \leq 2^{-2n} \sum_{1 \leq j \leq n} (2^j / \log j L_2 j).$$

It follows obviously that:

$$\lim_{k \to \infty} k^{-2} \sum_{1 \leq j \leq k} \mathbb{E}\|X_j\|^2 = 0.$$ 

Now we notice that:

$$\forall k \in I(n), \sup (\mathbb{E} \left( \sum_{1 \leq j \leq \alpha(n)} a_j \xi^j_k (2^n / \log n L_2 n), \sum_{r \geq 1} |a_r| \leq 1 \right) = (2^n / \log n L_2 n) \sup (\sum_{1 \leq j \leq \alpha(n)} a_j^2, \sum_{j \geq 1} |a_j| \leq 1) \leq (2^n / \log n L_2 n).$$

It follows that:

$$\Lambda(n) \leq 1 / \log n L_2 n,$$

for every $n$, and also:

$$\forall \epsilon > 0, \sum_{n \geq 1} \exp(-\epsilon / \Lambda(n)) < + \infty.$$
So all the hypotheses of Theorem 3 are fulfilled.

Now we will see that the SLLN fails for the sequence \((X_n)\). To see this it suffices to prove:

\[
(4) \quad \sum_{n \geq 1} P(2^{-n} \left\| \sum_{k \in I(n)} X_k \right\| > 1) = + \infty.
\]

Notice first that:

\[
P(2^{-n} \left\| \sum_{k \in I(n)} X_k \right\| > 1) = 1 - (1 - P(2^{-n/2} \left| \sum_{1 \leq i < 2^n} \epsilon_i \right| > (\log n 2^{n/2})^{1/2}))^{\alpha(n)},
\]

By Kolmogorov's converse exponential inequality (see [25] Theorem 5.2.2.) one has for \(n\) large enough:

\[
P(2^{-n} \left\| \sum_{k \in I(n)} X_k \right\| > 1) \geq 1 - (1 - 2n^{-L_2} \alpha(n))^{\alpha(n)},
\]

and also for \(n\) large enough:

\[
P(2^{-n} \left\| \sum_{k \in I(n)} X_k \right\| > 1) \geq 1/n,
\]

and (4) follows.

So Theorem 3 doesn't extend to \(c_0\). The main reason for this seems to be the fact that \(c_0\) is not a type 2 space; so in \(c_0\) hypothesis (ii) of Theorem 3 doesn't imply that \((X_n)\) \(\in\) WLLN. In our example it is easy to check that \((X_n)\) doesn't satisfy the WLLN.

The problem of finding extensions of Theorem 3 can now be set more precisely:

- Does it exist type 2 spaces which aren't 2-uniformly smooth in which Theorem 3 remains true?
- Does it exist non type 2 spaces in which the conclusion of Theorem 3 becomes true if one adds the complementary hypothesis \((X_n)\) \(\in\) WLLN?

Another approach of the Kolmogorov and Prohorov SLLN is to consider only very special 2-uniformly Banach spaces, in order to obtain results under simpler
The following hypotheses. For instance, we will be concerned in the next section by what happens in Hilbert spaces.

4. SOME SLLN IN HILBERT SPACES.

In the special case when \((B, \| \cdot \|)\) is a real separable Hilbert space, both the Kolmogorov and the Prohorov SLLN can be stated in the following nice ways:

**COROLLARY 1:** Let \((X_n)\) be a sequence of centered, independent r.v. with values in a real separable Hilbert space \((H, <, >)\). Suppose that the following 2 properties hold:

a) \(\exists M > 0 : \forall k \in \mathbb{N} \quad \|X_k\| \leq M \left( k / (L \cdot k)^{\frac{3}{2}} \right) \) a.s.

b) \(\sum_{j \geq 1} \sup \left( \mathbb{E} <f,X_j>^2 / j^2 , \|f\|_H \leq 1 \right) < \infty \).

Then:

\((X_n) \in \text{WLLN} \Leftrightarrow (X_n) \in \text{SLLN}\).

**COROLLARY 2:** Let \((X_n)\) be a sequence of centered, independent r.v. with values in a real separable Hilbert space \((H, <, >)\). Suppose that the following hold:

a) \(\exists M > 0 : \forall k \in \mathbb{N} \quad \|X_k\| \leq M \left( k / L \cdot k \right) \) a.s.

b) \(\forall \in > 0, \quad \sum_{n \geq 1} \exp(-\in / \Lambda(n)) < \infty \).

where for every integer \(n\):

\[ \Lambda(n) = 2^{-2n} \sum_{j \in I(n)} \sup \left( \mathbb{E} <f,X_j>^2 , \|f\|_H \leq 1 \right). \]

Then:

\((X_n) \in \text{WLLN} \Leftrightarrow (X_n) \in \text{SLLN}\).
For proving these 2 results, as previously it suffices to consider the symmetrical case; moreover the 2 proofs reduce to show the implication:

\[(X_n) \in \text{WLLN} \Rightarrow n^{-2} \sum_{1 \leq k \leq n} \|X_k\|^2 \overset{P}{\to} 0.\]

This follows easily from the fact that a Hilbert space has cotype 2 (see [11] for the details).

A natural question at this stage is the following one: "Would an hypothesis weaker than \((X_n) \in \text{WLLN}\) (of course added to a) and b) be sufficient to ensure that \((X_n) \in \text{SLLN}\)?" We will discuss this problem in a few words and show that in general it is not possible to weaken the condition \((X_n) \in \text{WLLN}\). For simplicity, we suppose till the end of this section that the \(X_n\) are symmetrically distributed.

Under the boundedness assumption a) of Corollary 1 or Corollary 2 it is clear that \(S_n/n\) is pregaussian (by Theorem 3.5 of [13]). For every \(n\) we denote by \(\mu_n\) a gaussian measure on \((\mathcal{H}, \mathcal{B})\) which has the same covariance structure as \(S_n/n\). A Hilbert space being both of type 2 and of cotype 2, it is easy to see that:

\[(X_n) \in \text{WLLN} \Rightarrow \mu_n \overset{w}{\to} \delta_0.

It is well known that this weak convergence of gaussian measures can be expressed in terms of weak integrability properties of the \(S_n/n\). Let's recall how it works. For every \(n\) we denote by \((\alpha_{n}^k)\) the sequence of the eigenvalues of the covariance operators \(A_n\) of the measures \(\mu_n\); these covariance operators being associated with gaussian measures one has:

\[\forall n, \forall k, \quad \alpha_{n}^k \geq 0,\]

\[\forall n, \quad \sum_{k \geq 1} \alpha_{n}^k < +\infty.\]

Moreover:

\[\mu_n \overset{w}{\to} \delta_0 \Leftrightarrow \lim_{n \to \infty} \sum_{k \geq 1} \alpha_{n}^k = 0 \quad (5).\]
This condition is a weak integrability condition because \( \alpha_n^k = E \langle \varepsilon, S_n/n \rangle^2 \) for a suitable \( \varepsilon \); but (5) is not implied in general by an hypothesis like b) in Corollary 1 or Corollary 2. So the assumption \( (X_n) \in \text{WLLN} \) cannot be dropped in general.

If the reader prefers statements in "gaussian language" he can replace the assumption " \( (X_n) \in \text{WLLN} \) " by the condition (5) rewritten in the following form:

"The covariance operator \( A_n \) of the r.v. \( S_n/n \) converge in the nuclear norm to the covariance operator of the degenerated gaussian measure \( \delta_0 \)."

As a conclusion to this paper, we will give another application of Theorem 2.

4. APPENDIX : THE LAW OF THE ITERATED LOGARITHM IN 2-UNIFORMLY SMOOTH BANACH SPACES.

The law of the iterated logarithm (LIL) is usually considered as a refinement of the SLLN, so it is natural to look what Theorem 2 brings for its study in 2-uniformly smooth Banach spaces.

First we recall shortly the 2 forms taken by the LIL in the infinite dimensional setting.

Let \( X \) be a r.v. which takes its values in a real separable Banach space \( (B, \| \|) \). Let \( (X_n) \) denote a sequence of independent copies of \( X \); for every \( n \), we put as usual:

\[ S_n = X_1 + \ldots + X_n. \]

- One says that \( X \) satisfies the bounded LIL \( (X \in \text{BLIL}) \) if and only if:
\[ P(\sup_n \| S_n / a_n \| < + \infty ) = 1 , \]

where:
\[ a_n = (2nL_2 n)^{1/2} . \]

One says that \( X \) satisfies the compact LIL (\( X \in CLIL \)) if and only if there exists a set \( K \subset B \), which is compact, convex and symmetric, such that:

1) \[ P( \lim_{n \to + \infty} d(S_n / a_n, K) = 0 ) = 1 , \]

where:
\[ d(x, K) = \inf \left\{ d(x, y) : y \in K \right\} . \]

2) \[ P( C(S_n / a_n) = K ) = 1 , \]

where \( C(\alpha_n) \) denotes the cluster set of the sequence \( (\alpha_n) \).

It is well known [16] that the set \( K \) is completely defined by the covariance operator of \( X \).

Necessary and sufficient conditions for the 2 forms of the LIL in a 2-uniformly smooth Banach space have been discovered recently by M. Ledoux [19]; his method of proof extends the one used in Hilbert spaces by V. Goodman, J. Kuelbs and J. Zinn [9]. M. Ledoux's result is as follows:

**THEOREM 4:** Let \( X \) be a r.v. with values in a 2-uniformly smooth Banach space \(( B, \| \| ) \). Then:

1) \( X \in BLIL \) if and only if
   \[ \forall f \in B', \ E f(X) = 0, \ E f^2(X) < + \infty . \]

2) \( X \in CLIL \) if and only if
   \[ \forall f \in B', \ E f(X) = 0, \ E f^2(X) < + \infty . \]

i) \( X \in BLIL \).

ii) The covariance operator of \( X \) is compact.
Here our goal will not be to give a complete proof of Theorem 4. We want only to show to the reader that by applying Theorem 2, the difficult part of M. Ledoux's direct proof can be made very short and very simple. For the reader's convenience, we recall first the classical reductions in proving a LIL result like Theorem 4.

The necessary parts of 1) and 2) are well known; they are true in every Banach space [17]. By a classical closed graph argument [23] we need only to prove the sufficiency in part 1). Another classical argument [23] shows that it suffices to consider the symmetrical case. Moreover by the Borel Cantelli Lemma the proof finally reduces to show the following 2 properties:

\[ \exists \epsilon > 0 : \]

(i) \[ \sum_{k \in I(n)} P( \| \sum_{k \in I(n)} u_k \| > \epsilon (2^n \log n)^{1/2} ) < + \infty ; \]

(ii) \[ \sum_{k \in I(n)} P( \| \sum_{k \in I(n)} v_k \| > \epsilon (2^n \log n)^{3/2} ) < + \infty ; \]

where for any \( k \in I(n) \):

\[ u_k = X_k \cdot (2^n / \log n)^{1/2} < \| X_k \| \leq (2^n / \log n)^{1/2} \]

\[ v_k = X_k \cdot (\| X_k \| \leq (2^n / \log n)^{3/2}) \]

Now we will show that property (ii) can be obtained by an application of Theorem 2.

Without loss of generality, we can assume:

\[ \sup \{ \| EF^2(x) \|_{B^*}, \| \| \leq 1 \} \leq 1. \]

Let \( \epsilon > 0 \) be arbitrary; as we have done previously in the proof of Theorem 3, we will bound \( P( \| \sum_{k \in I(n)} v_k \| > \epsilon (2^n \log n)^{3/2} ) \) by:

\[ P( \| \sum_{k \in I(n)} v_k \| > 2^{n-1} \log n \epsilon^2 / C ) + P( \sum_{j \in I(n)} P(\{ s_{j-1}(v) : v_j > 16 \} ) , \]

where:
By applying a SLLN of W. Feller [7], one obtains:

$$\forall \epsilon > 0 \quad \sum_{n \geq 1} \mathbb{P}(\sum_{k \in I(n)} \|v_k\|^2 > 2^{n-1} \log n \epsilon^2 / C) < +\infty.$$  

By choosing $\epsilon = 2^{9/2} e^{4} C^{1/2}$, and by applying Theorem 2 for $K = 16$, one gets for $n$ large enough:

$$\mathbb{P}(\sum_{k \in I(n)} F(S_{j-1}(Y))(Y_j) > 16C) \leq 4 (n \log 2)^{-3/2},$$

and so property (ii) holds with $\epsilon = 2^{9/2} e^{4} C^{1/2}$.

Property (i) also holds for this value of $\epsilon$ (in fact it holds for every $\epsilon > 0$, see [19]).

It appears from this short computation that Theorem 2 is also a very good tool for the study of the LIL in a 2-uniformly smooth Banach space.

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ON SOME ERGODIC THEOREMS FOR VON NEUMANN ALGEBRAS

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The aim of this article is to extend two ergodic theorems of Ryll-Nardzewski to the von Neumann algebra set-up. In the classical case, for a given probability space \((\Omega, B, m)\) and a measurable, measure-preserving transformation \(T: \Omega \rightarrow \Omega\), in the individual ergodic theorem one considers the almost everywhere convergence of the averages

\[
\frac{1}{n} \sum_{k=0}^{n-1} f(T^k \omega)
\]

where \(f \in L^1(\Omega, B, m)\). Following Brunel and Keane ([1]) in [7] Ryll-Nardzewski deals with the averages of the form

\[
\frac{1}{n} \sum_{j=0}^{n-1} f(T^{k_j} \omega)
\]

where \(k_0 < k_1 < ...\) is an increasing subsequence of the sequence of non-negative integer indices. In this paper the problem of convergence of the above sums has been reduced to the problem of classical averages with "weights"

\[
\frac{1}{n} \sum_{j=0}^{n-1} \beta_j f(T^j \omega)
\]

Passing to the non-commutative case, we begin with some notations and definitions.

Let \(A\) be a semifinite von Neumann algebra with the separable predual \(A_*\) and let \(\rho\) be a faithful normal state on \(A\). Let \(\alpha: A \rightarrow A\) be an ultra-weakly continuous positive contraction with \(\alpha 1 = 1\) (1 denotes here the identity in \(A\)) and \(\rho(\alpha(x)) \leq \rho(x)\) for \(x \in A^+\).

In this context, the usual ergodic averages appear as the sums

\[
\frac{1}{n} \sum_{j=0}^{n-1} \alpha^j(a)
\]

where \(a \in A\).

Recently, Petz ([6]) proved another non-commutative version of the individual ergodic theorem. This theorem asserts that under the above conditions concerning the von Neumann algebra \(A\), the transformation \(\alpha\) and the state \(\rho\), ergodic averages (1) converge quasi-uniformly to
some $\hat{a} \in \hat{A}$ for each $a \in A$. A sequence $(a^n) \subseteq A$ is said to converge to some $a \in A$ quasi-uniformly in $A$ if, for each non-zero projection $f \in A$, there exists a non-zero projection $e \leq f$ in $A$ such that $\| (a^n - a)e \| \to 0$ as $n \to \infty$.

In this article we shall consider the quasi-uniform convergence of averages with weights

$$\frac{1}{n} \sum_{j=0}^{n-1} \beta_j a^j(a). \quad (2)$$

It can be shown that, just as in the classical case, the limit of ergodic averages with weights $(\beta_j)$ exists if $(\beta_j)$ is a bounded Besicovitsch sequence, i.e. the sequence of complex numbers $(\beta_j)$ with the properties

1° $|\beta_j| \leq \text{const}$

2° for each $\varepsilon > 0$, there exists a trigonometric polynomial

$$w_{\varepsilon}(j) = \sum_{s} \gamma_s e^{i\theta_s j}$$

such that

$$\lim sup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |\beta_j - w_{\varepsilon}(j)| \leq \varepsilon.$$

Then we can formulate

**THEOREM 1.** For each $a \in A$ and each bounded Besicovitsch sequence $(\beta_j)$, the sequence of ergodic averages of the form

$$\frac{1}{n} \sum_{j=0}^{n-1} \beta_j a^j(a) \quad (3)$$

converges quasi-uniformly to some $\hat{a} \in A$.

**SKETCH of PROOF.** Let $K$ be a unit circle with the normalized Lebesgue measure $\lambda$. Consider the von Neumann algebra $\tilde{A} = L_\omega(K, \lambda) \otimes A = L_\omega(K, \lambda; A)$ which consists of all essentially bounded ultra-weakly measurable functions $f : K \to A$ with the norm $\| f \|_\omega = \sup \text{ess sup} \| f(z) \|_{z \in K}$ (cf. [8], p.68).

Let $\tilde{\rho}$ denote a tensor product state $\rho \otimes \lambda$ given by the formula

$$\tilde{\rho}(f) = \int_K \rho(f(z)) \lambda(dz) \quad \text{for} \quad f \in A.$$

Now, for fixed $\theta \in [0, 2\pi)$, we construct a transformation $\psi : \tilde{A} \to \tilde{A}$
given by \((\psi g)(z) = \alpha(g e^{i\theta} z)\) for \(z \in K\) and \(g \in A\). Applying the ergodic theorem of Petz to \(A, \hat{\nu}\) and \(\psi\), we obtain that, for each \(g \in \hat{A}\), the averages \(\frac{1}{n} \sum_{j=0}^{n-1} \psi^j g\) converge quasi-uniformly in \(\hat{A}\) to some \(g \in \hat{A}\). Now, using the very useful lemma of Ngiem Dang-Ngoc [2], we get that \(\frac{1}{n} \sum_{j=0}^{n-1} (\psi^j g)(z)\) tends quasi-uniformly in \(A\) to \(g(z)\) for almost all \(z \in K\). Finally, putting \(g(z) = za\) for \(z \in K\) and \(a \in A\), we have that \(\frac{1}{n} \sum_{j=0}^{n-1} e^{i\theta^j} a^j(a)\) converges quasi-uniformly in \(A\).

In this way, we have obtained the theorem for the special case of Besicovitsch sequences, namely, \(B_j = e^{i\theta^j}\). A suitable approximation ends the proof.

REMARK. The first generalization of the individual ergodic theorem to the von Neumann algebra set-up was given by Lance [4]. The Lance theorem states that, under a little stronger assumptions, sums (1) converge \(\rho\)-almost uniformly in \(A\). Recall that \((a_n) \subseteq A\) is said to converge \(\rho\)-almost uniformly in \(A\) to some \(a \in A\) if, for each \(\varepsilon > 0\), there exists a projection \(e\) in \(A\) such that \(\rho(e) \geq 1 - \varepsilon\) and \(\|(a_n - a)e\| \to 0\) as \(n \to \infty\). In the paper of Ngiem Dang-Ngoc [2] the \(\rho\)-almost uniform convergence was used. Recently, in [5] Paszkiewicz proved that the two kinds of convergence, namely, \(\rho\)-almost uniform and quasi-uniform, are equivalent for bounded sequences of operators from \(A\).

In the sequel, just as in the classical case, a transformation \(\alpha : A \to A\) is said to be ergodic if the equation \(\alpha x = x\) has only the solution of the form \(\lambda l\) in \(A\). Analogously, \(\alpha\) is said to be a weak mixing if it is an ergodic transformation and has no eigenvalues \(\lambda \neq 1\) (cf. [3]).

Now, assume additionally that \(\alpha : A \to A\) is an isometry and characterize transformations \(\alpha\) for which all the limits of form (2) are constant (i.e. of the form \(\lambda l\)).

THEOREM 2. Ergodic averages (2) weighted by means of bounded Besicovitsch sequences are constant if and only if the transformation \(\alpha\) is a weak mixing.

SKETCH OF PROOF. NECESSITY. Putting \(B_j = 1\), we get
\[ \frac{1}{n} n^{-1} \sum_{j=0}^{n-1} a_j(a) \rightarrow c(a) \text{ quasi-uniformly as } n \rightarrow \infty. \] Assume that \( \alpha \alpha = \alpha. \)

Then \[ \frac{1}{n} n^{-1} \sum_{j=0}^{n-1} a_j(a) = \alpha; \text{ thus } \alpha = c(a) 1. \]

Assume now that \( \alpha \alpha = \lambda \alpha \) and \( \lambda \neq 1. \) Then \( \lambda = e^{-i\theta} \) for some \( \theta \in (0,2\pi). \) The averages

\[ (4) \quad \frac{1}{n} n^{-1} \sum_{j=0}^{n-1} e^{i\theta j} a_j(a), \]

being equal to \( \alpha, \) have a limit of the form \( c_\theta(a) 1. \) Thus \( \alpha = c_\theta(a) 1. \)

On the other hand, sums (4) can be written in the form

\[ \frac{1}{n} n^{-1} n^{-2} e^{i\theta j} a_j(a) \]

and, therefore, the limit is \( \lambda^{-1} a(c_\theta(a) 1). \) Thus \( \lambda^{-1} c_\theta(a) \alpha 1 = c_\theta(a) 1 \) and, finally, \( c_\theta(a) = 0 \) and \( \alpha = 0. \)

**SUFFICIENCY.** Let us remark that, by the ergodicity of \( \alpha, \) sums (2) with \( \beta_j = 1 \) tend to constants. If \( \beta_j = e^{i\theta j} \) for some \( \theta \in (0,2\pi), \) averages (4) converge to \( \hat{\alpha} = \lambda^{-1} a(\hat{\alpha}) \) for some \( \hat{\alpha} \in A, \)

where \( \lambda = e^{-i\theta} \neq 1. \) Thus, by the assumption of the weak mixing, we get \( \hat{\alpha} = 0. \)

If \( \beta_j = w(j) \) where \( w \) is a trigonometric polynomial, then the averages weighted by means of the sequence \( \beta_j \) obviously tend to constants. An approximation completes the proof.

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LOG LOG LAW FOR GAUSSIAN RANDOM VARIABLES IN ORLICZ SPACES

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1. Introduction. In the theory of Gaussian processes the concept of reproducing kernel space is very important. It is especially useful for processes with a.e. paths in infinite dimensional spaces. Using this notion Le Page (1973) proved the law of the iterated logarithm for Gaussian processes with almost all sample paths in Banach spaces. He generalized the classical result of Hartman and Wintner (1941) for normal distributed $\mathbb{R}$-valued random variable using Strassen's result (1964) for $\mathbb{R}^k$-valued random vectors.

However, construction of an analogue of reproducing kernel Hilbert space and abstract Wiener space for a given Gaussian measure on non-locally convex space remains an open question. It was done by Lawniczak (1982) for Orlicz spaces by means of quasi-additive measurable functionals. In present paper we do it in another way.

Moreover, for symmetric Gaussian measure concentrated on Orlicz space with topology generated by a $p$-homogeneous seminorm ($0 < p \leq 1$) we prove the law of the iterated logarithm.

2. Let $(T, \mathcal{F}, m)$ be a positive $\sigma$-finite separable measure space. Let $\phi$ be a Young function, i.e. a continuous non-decreasing function defined for $u \geq 0$ and such that $\phi(u) = 0$ if and only if $u = 0$. Assume that $\phi$ satisfies $\Delta_2$-condition i.e. $\phi(2u) \leq C\phi(u)$ for some $C > 0$ and every $u > 0$. For a measurable function $x : T \to \mathbb{R}$ put

$$\mathcal{L}_\phi \{ x \} = \int_T \phi(|x(t)|) \, dm(t),$$

$$\| x \|_\phi = \inf \left\{ u > 0 : \mathcal{L}_\phi \{ \frac{x}{u} \} \leq 1 \right\},$$

and denote by $\mathcal{L}_\phi$ the collection of all $\mathcal{F}$-measurable functions $x$ with $\mathcal{L}_\phi (x) < \infty$. Let $L_\phi$ be the space of all equivalence classes of functions from $\mathcal{L}_\phi$ which are equal a.e. $[m]$. Then $L_\phi$ is a vector space and $\| \cdot \|_\phi$ is a (usually non-homogeneous) seminorm on $L_\phi$. Moreover $(L_\phi, \| \cdot \|_\phi)$ is a complete separable metric space called Orlicz space.
Let $X$ be a symmetric Gaussian random variable with values in $L^q$. Then (Th. 1.1 [3]) there exists a measurable Gaussian random process $\xi$ with almost all paths in $L^q$ such that $\tilde{\xi} = X$ a.s., where $\tilde{\xi}(\omega)$ denotes the equivalence class of $\mathcal{F}$-measurable functions corresponding to $\xi(\omega, \cdot)$. The measure induced on $L^q$ by $\xi$ coincides with the distribution of $X$.

Consider the reproducing kernel Hilbert space $H_R$ generated by the covariance function $R = R(s,t)$ of $\xi$ (see [1]), where $\xi$ is a Gaussian process corresponding to the symmetric Gaussian random variable $X$. Because of measurability of $\xi$ the space $H_R$ is separable.

Denote $(\cdot, \cdot)_R$ and $\| \cdot \|_R$ the inner product and the Hilbert norm, respectively, in $H_R$.

**Lemma 1.** The space $H_R$ is contained in $L^q$.

**Proof.** For $h \in H_R$ we have

$$|h(s)| = |(h, R(s, \cdot))_R| \leq \|h\|_R R(s, s)^{1/2}. \tag{1}$$

Since $R(\cdot, \cdot)^{1/2} \in L^q$ (Prop. 1 [4]) it follows that $\gamma_R(h) < \infty$.

**Lemma 2.** Let $i: H_R \to L^q$, $i(h) = [h]$, where $[h]$ denotes the equivalence class of all $\mathcal{F}$-measurable functions which are a.e. equal to $h$. Then $i$ is continuous.

**Proof.** If $h_n \to 0$ in $H_R$, then $h_n(s) = (h_n, R(s, \cdot))_R \to 0$ and by (1) $h_n(s) \leq C R(s, s)^{1/2}$ for every $s \in \mathcal{T}$, where $C = \sup_n \|h_n\| < \infty$. Let $a > 0$.

By the Lebesgue Dominated Convergence Theorem $\gamma_R(a h_n) \to 0$ i.e. $h_n \to 0$ in $L^q$.

Let now $\text{Ker} i = \{h \in H_R : [h] = 0\}$. Define $H = (\text{Ker} i)^\perp \subset H_R$. $H$ is a separable Hilbert space which may be identified with a subset of $L^q$.

Let $(a_n)$ be a CONS in $H$. Then (Th. 2 [4]) there exists a sequence $(\lambda_n)$ of independent normally distributed real random variables with mean zero and variance 1 such that the series $\sum a_n \lambda_n$ converges a.s. in $L^q$. If $\mu$ denotes the distribution of this series and $\mu_X$ the distribution of Gaussian random variable $X$, then $\mu = \mu_X$ and $H$ is dense in supp $\mu$.

Let $K = \{h \in H : \|h\| \leq 1\}$ be the unit ball of $H$.

**Lemma 3.** The unit ball $K$ of $H$ is compact in $L^q$.

**Proof.** Because of separability of $L^q$ it suffices to prove the sequential compactness. Suppose, on the contrary, that there exists $\epsilon > 0$ and
a sequence \((h_n) \subset K\) such that for every \(k, n \in \mathbb{N}\)

\[
|h_n - h_k|_\Phi = \inf \left\{ u > 0 : \int_\mathcal{F} \left( \frac{|h_n - h_k|}{u} \right) \, dm \leq u \right\} > \varepsilon
\]

Hence for every \(k, n \in \mathbb{N}\)

\[
(2) \quad \int_\mathcal{F} \left( \frac{|h_n - h_k|}{\varepsilon} \right) \, dm > \varepsilon.
\]

Since \(K\) is weak compact as a unit ball in a Hilbert space there exists a subsequence \((h_{n_k}) \subset (h_n)\) such that the sequence \(h_{n_k}(s) = (h_{n_k}, R(s, \cdot))_\mathcal{F}\) is convergent for every \(s \in \mathcal{T}\). From (1) it follows that for every \(s \in \mathcal{T}\)

\[
\left| \frac{h_{n_k}(s) - h_{n_k+1}(s)}{\varepsilon} \right| \leq \frac{2}{\varepsilon} R(s, s)^{1/2}
\]

and by the Lebesgue Dominated Convergence Theorem we have

\[
\int_\mathcal{F} \left( \frac{|h_{n_k} - h_{n_k+1}|}{\varepsilon} \right) \, dm \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

This contradicts (2) and hence \(K\) is relatively compact. Now let \(K \ni h_n \rightarrow h\) in \(L_\Phi\) norm. We take a weakly convergent subsequence \((h_{n_k})\) of \((h_n)\) and let \(h_0 \in K\) be its weak limit. By (1) and again Lebesgue’s Dominated Convergence Theorem we have \(h_{n_k} \rightarrow h_0\) in \(L_\Phi\) and \(h = h_0\).

Hence \(K\) is closed and Lemma is proved.

3. Now we consider Orlicz spaces \(L_\Phi\) for which there exists \(p\)-homogeneous measurable seminorm \(|\cdot|_\Phi, 0 < p \leq 1\), equivalent to \(|\cdot|_\Phi^p\). Such condition holds for instance (see [10]) if \(\Phi\) is a \(p\)-convex Young function, i.e., \(\Phi(at + bs) \leq a^p \Phi(t) + b^p \Phi(s)\) for all \(a, b > 0, a + b \leq 1\) and for all \(t, s \geq 0\). It also holds, if \(\Phi(t) = \Phi_0(t^r)\), where \(\Phi_0\) is a convex Young function, \(0 < r < \infty\). In these two cases the resulting \(p\)-homogeneous seminorm is \(\|\cdot\|_\Phi^p\), where

\[
\|x\|_\Phi^p = \inf \left\{ u > 0 : \int_\mathcal{F} \left( \frac{|x|}{u} \right) \, dm \leq 1 \right\}, \quad x \in L_\Phi.
\]

The considered class of Orlicz spaces contains many spaces which are neither Banach nor even locally convex spaces, for example \(L_p\) spaces, \(0 < p < 1\).

Let now \(X\) be a symmetric Gaussian random variable with values in \(L_\Phi\). Consider the following expansion of \(X\) in \(L_\Phi\):

\[
X = \sum_{n=1}^\infty a_n \lambda_n,
\]

where \((a_n)\) is an orthonormal basis in the reproducing kernel space.
$H \subset L^q$, $(\lambda_n)$ is an appropriate standard normal sequence and the series converges a.s. in $L^q$.

Denote
$$X(k) = \sum_{n=1}^{k} a_n \lambda_n.$$

For the sequence $(X_n)$ of independent copies of $X$ we define
$$\zeta_n = \frac{X_1 + \cdots + X_n}{(2n \log \log n)^{1/2}}.$$

Our main result is as follows:

**Theorem.** The sequence $(\zeta_n)$ is relatively compact in $L^q$ with probability 1 and the set of its limit points coincides with the unit ball $K$ of $H$.

We prove this theorem using similar arguments as in the proof of the law of the iterated logarithm for Gaussian processes with sample paths in Banach spaces [9]. To prove this we need the following modification of Fernique's estimate known, at present, only for $p$-homogeneous seminorms:

**Lemma 4.** ([5], [10]). Let $(E, \mathcal{B})$ be a measurable linear space, let $X$ be a symmetric Gaussian random variable with values in $E$ and let $|\cdot|$ be a measurable $p$-homogeneous seminorm in $E$, $0 < p \leq 1$. Then

$$\log \frac{\beta |x|^2}{4s_{\beta}^2} < \infty$$

for $\beta < \frac{\log \frac{x}{2s_{\beta}}}{4s_{\beta}^2} \left(2p/2 - 1\right)^{2/p}$, where $P\{|X| \leq s\} = r > \frac{1}{2}$.

Proofs of next two Propositions are modifications of those in [9].

**Proposition 1.** For $\varepsilon > 0$, $r \sim 1$

$$P\left\{ \frac{X}{(2 \log \log n)^{1/2}} \notin K_{\varepsilon} \right\} \leq e^{-r^2 \log \log n}$$

for all sufficiently large $n$.

**Proof.** Suppose $\varepsilon > 0$, $r > 1$, $k > 1$, $n > 3$. For $r > r_0$,

$$P\left\{ \frac{X}{(2 \log \log n)^{1/2}} \notin K_{\varepsilon} \right\} \leq P\left\{ \frac{X(k)}{r_0(2 \log \log n)^{1/2}} \notin K \right\} +$$
The first term on the right side of (4) is equal to $P\{X^2 > 2r^2 \log \log n\}$, where $X^2_k$ is a chi-square random variable with $k$ degrees of freedom, so for all sufficiently large $n$ and for $r < r_0$ this is less than $\frac{1}{2} \exp(-r^2 \log \log n)$. To estimate the second term let $X(k)/r_0(2 \log \log n)^{1/2} = h \in K$ and $\|h - X/(2 \log \log n)^{1/2}\| > \epsilon$. Since $K$ is compact in $L\Phi$ and the scalar multiplication is uniformly continuous on compact sets we can find $r_0 \sim 1$ such that $\| (r_0 - 1) h \| < \epsilon/2$ a.e. Hence and by Lemma 4 we have

$$P\{h \in K, \|h - \frac{X}{(2 \log \log n)^{1/2}}\| > \epsilon\} \leq P\{\|X - X(k)\| > \frac{\epsilon}{2}(2 \log \log n)^{p/2}\} \leq \exp(-\alpha(\frac{\epsilon}{2})^2/p) \cdot \exp \alpha \|X - X(k)\|^{2/p}$$

and the latter integral is finite for

$$\alpha < \frac{(e^{p/2} - 1)^2/p}{4t^2/p} \log \frac{P\{\|X - X(k)\| \leq t\}}{P\{\|X - X(k)\| > t\}},$$

where $P\{\|X - X(k)\| \leq t\} > \frac{1}{2}$. Since $\|X - X(k)\| \to 0$ in probability we can choose such $\alpha$ and $k$ that $2\alpha(\frac{\epsilon}{2})^2/p > r^2$ and $\mathbb{E} \exp(\alpha \|X - X(k)\|^{2/p}) < \infty$.

For these $k$ and $\alpha$ the second term of (4) is estimated by $\frac{1}{2} e^{-r^2 \log \log n}$ for all sufficiently large $n$.

**Proposition 2.** For $\epsilon > 0$, $r > 1$ it is possible to choose $c > 1$ sufficiently close to one so that

$$P\left\{\max_{[c^n]_{i=1}^{c^{n+1}}} \| \mathbf{Z} - \mathbf{Z}^{[c^n]} \| > \epsilon \right\} \leq e^{-r^2 \log \log [c^n]}$$

for all sufficiently large $n$, where $[a]$ denotes the integer part of $a$.

Using these Propositions and standard arguments we can prove our Theorem.

**Remark.** If the space did not have an equivalent $p$-homogeneous seminorm then our Theorem would hold if Conjecture 2 in[5] were true with $\alpha(\epsilon)$ going to infinity for $\mu_{X-X(k)} \Rightarrow \delta_0$. One can easily see that for the space $L_0[0,1]$ with usual seminorm this Conjecture holds.
References


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A FEW REMARKS ON THE ALMOST UNIFORM ERGODIC THEOREMS
IN VON NEUMANN ALGEBRAS

R. Jajte (Kodz)

Introduction. Let \((\Omega, \mathcal{F}, \mu)\) be a probability space. Then the commutative von Neumann algebra \(\mathcal{A} = L_\infty(\Omega, \mathcal{F}, \mu)\) has a faithful tracial state given by the integral \(\tau_\mu(t) = \int_\Omega f \, d\mu\). By Egoroff's theorem, the almost sure convergence of a sequence \(\{\xi_n\}\) of random variables is equivalent to the uniform convergence of \(\{\xi_n\}\) on measurable sets of measure arbitrarily close to one (almost uniform convergence). This fact makes possible to express the almost sure convergence purely in terms of the algebra \(\mathcal{A}\) (at least for sequences of bounded functions), without any reference to the base space \(\Omega\). Namely, we may restate the almost sure convergence by means of \(L_\infty\)-norm, state \(\tau_\mu\) and the characteristic functions (of "large" measurable subsets). This suggests the following definition of almost uniform convergence in a (non-commutative, in general) von Neumann algebra \(\mathcal{A}\).

Let \(\phi\) be a faithful normal state on \(\mathcal{A}\). We say that a sequence \(\{x_n\}\) of elements of \(\mathcal{A}\) converges almost uniformly to \(x \in \mathcal{A}\) if for each \(\varepsilon > 0\), there exists a projection \(p \in \mathcal{A}\) such that \(\phi(p) \geq 1 - \varepsilon\) and \(\|(x_n - x)p\| \to 0\) as \(n \to \infty\). In the commutative case of the algebra \(L_\infty(\Omega, \mathcal{F}, \mu)\) (over a probability space \((\Omega, \mathcal{F}, \mu)\)), the almost uniform convergence coincides with the almost everywhere convergence. Thus it is clear that many problems from classical ergodic theory can be formulated in the commutative von Neumann algebras \(L_\infty\). The present paper concerns itself with certain ergodic type theorems in general (not necessarily commutative) von Neumann algebras. Recently, E.C. Lance [22] and Y.G. Sinai and V.V. Anshelevich [31] have shown non-commutative analogues of Birkhoff individual ergodic theorem for automorphisms of operator algebras.

THEOREM OF LANCE [22]. Let \(\alpha\) be a \(*\)-automorphism of a von Neumann algebra \(\mathcal{A}\), and let \(\phi\) be a faithful normal and \(\alpha\)-invariant state of \(\mathcal{A}\). Then, for any \(x \in \mathcal{A}\), the averages
\[
s_n(x) = n^{-1} \sum_{k=0}^{n-1} \alpha^k x
\]
converge almost uniformly to some \(\hat{x} \in \mathcal{A}\).
The generalizations of Lance's result were proved by J.P. Conze and N. Dang Ngoc [4], B. Kümmener [21], F.J. Yeadon [34], M. Goldstein [12], S. Watanabe [33], D. Petz [26]; comp. also [20]. The proof of Lance's result (and its generalizations) are rather long and most of them are based on the advanced theory of functions on convex sets. The most elementary and direct proof was given by M. Goldstein [12].

In the next section we shall prove the generalization of Lance's result given by B. Kümmener [21] (for positive maps and subinvariant states). The Kümmener's proof is based on the advanced theory of convex functions. Our proof will follow in part the ideas of M. Goldstein [12]. At the same time it can be treated as the proof of Lance's theorem, and it seems to be shorter and more elementary in comparison with the original proof of Lance [22].

1. Let us begin with some notation and preliminaries. In the sequel, we shall assume the familiarity with the basic theory of von Neumann algebras as contained in the first chapters of [6], [29] or [32]. Throughout, $\mathcal{A}$ will denote a semifinite von Neumann algebra with a faithful normal state $\phi$. The norm in $\mathcal{A}$ will be denoted by $\|\cdot\|_\infty$. The symbol $\text{Proj } \mathcal{A}$ will stand for the lattice of all projectors in $\mathcal{A}$. Always, $p^\perp = 1 - p$ for $p \in \text{Proj } \mathcal{A}$. For a self-adjoint operator $\xi \in \mathcal{A}$, we shall denote by $e_\xi(\cdot)$ its spectral measure, i.e. $\xi = \int \lambda e_\xi(d\lambda)$. $\mathcal{A}'$ will always denote the commutant of $\mathcal{A}$. Let us collect here a few facts that we shall need later.

1. Let $\psi \in \mathcal{A}^*$ and $\psi(x) \leq (x\xi, \xi)$ for some $\xi \in \mathcal{A}$ and all $x \in \mathcal{A}_+$, then there exists some $y \in \mathcal{A}_+$ (commutant of $\mathcal{A}$) such that $\psi(x) = (xy\xi, y\xi)$ for all $x \in \mathcal{A}$.

2. For $x \in \mathcal{A}$ with $\|x\|_\infty \leq 1$, and $q, r \in \text{Proj } \mathcal{A}$, if $\|q^\perp r\| < \alpha$ and $\|qx\| < \beta$, then $\|xr\|_\infty < \alpha + \beta$. To prove this, it suffices to estimate:

$$\|xr\xi\| \leq \|(q^\perp r)\|_\infty + \|(q)\|_\infty \|x\|_\infty \|r\|_\infty.$$

3. Let $p \in \text{Proj } \mathcal{A}$, $x \in \mathcal{A}$, and $\phi(p|x|^2p) < \varepsilon^4 < 1$. Then, putting $q = p e_{p|x|^2p} [0, \varepsilon^2]$, we have $q \leq p$, $\phi(p - q) \leq \varepsilon$ and $\|qx\| < \varepsilon$. We omit the easy and rather standard proof of (3). Now we are in a position to formulate and prove the Kümmener's result.

1. THEOREM OF KÜMMENER ([21], comp. also [12]). Let $\mathcal{A}$ be a von Neumann algebra with a faithful normal state $\phi$; let $\alpha$ be a normal positive linear map in $\mathcal{A}$ with $\alpha 1 \leq 1$ (1 denotes the identity in $\mathcal{A}$) and such that $\phi(\alpha x) \leq \phi(x)$ for all $x \in \mathcal{A}_+$. Then, for each $x \in \mathcal{A}$,
the ergodic averages

\[ s_n(x) = n^{-1} \sum_{k=0}^{n-1} \alpha^k x \]

converge almost uniformly in \( \mathcal{A} \) to an \( \alpha \)-invariant element \( \hat{x} \in \mathcal{A} \).

**Proof.** In our case the Gelfand-Naymark-Segal (GNS) representation of \( (\mathcal{A}, \phi) \) is faithful and normal, so, without any loss of generality, we can assume that \( \mathcal{A} \) acts in its GNS representation space \( \mathcal{H}_\phi \) in a standard way. In particular, we have \( \mathcal{H}_\phi = L^2(\mathcal{A}, \phi) \) - the completion of \( \mathcal{A} \) under the norm \( x \rightarrow \phi(x^*x)^{1/2} \), and \( \phi(x) = (x\xi, \xi) \) for \( x \in \mathcal{A} \), where \( \xi \) is a cyclic and separating vector in \( \mathcal{H}_\phi \). The norm in \( \mathcal{H}_\phi \) will be denoted by \( \| \cdot \|_2 \).

Let \( \mathcal{A}_S \) be the real linear space of all self-adjoint elements of \( \mathcal{A} \). \( L^2(\mathcal{A}_S, \phi) \) will stand for the completion of \( \mathcal{A}_S \) with respect to the norm \( x \rightarrow \phi(x^2)^{1/2} \). The main steps of the proof are the following:

(A) (Mean ergodic theorem). Under the assumptions of the theorem, for each \( x \in \mathcal{A} \), the ergodic means (1) go in \( L^2 \)-metric to an \( \alpha \)-invariant element \( \hat{x} \in \mathcal{A} \).

(B) \( L^2(\mathcal{A}_S', \phi) = \mathcal{H}_1 \oplus \mathcal{H}_2 \) where \( \mathcal{H}_1 = \{ y \in \mathcal{A}_S : y = ay \} \) and \( \mathcal{H}_2 = \{ y - ay ; y \in \mathcal{A}_S \} \). Here and elsewhere \( \{ z, \ldots \} \) denotes the closed linear subspace spanned by \( z \)'s.

(C) For each \( x \in \mathcal{A} \) and \( \varepsilon > 0 \) one can find four elements \( a_1, a_2, a_3, a_4 \) from \( \mathcal{A}_+ \) with \( \phi(a_i^2) < \varepsilon^2 \) (i = 1, 2, 3, 4) and some elements \( y_1, y_2, \ldots, y_M \) from \( \mathcal{A} \) such that \( x - \hat{x} = (a_1 - a_2) + i(a_3 - a_4) + \sum_{i=1}^{M} (y_i - ay_i) \), where \( \hat{x} \) is defined in (A).

(D) Under the assumptions about \( \alpha \) as in the theorem, there exists a linear positive map \( \alpha' : \mathcal{A}' \rightarrow \mathcal{A}' \) such that

(i) \( \alpha' \leq 1 \)

(ii) \( (\alpha'(y)\xi, \xi) \leq (y\xi, \xi) \) for \( 0 \leq y \in \mathcal{A}' \)

(iii) \( (\alpha(x)y\xi, \xi) \leq (x\alpha'(y)\xi, \xi) \) for all \( x \in \mathcal{A} \), \( y \in \mathcal{A}' \).

(E) Let \( \{ a_i \}_{i \in I} \) be a family of positive operators from \( \mathcal{A} \) and let \( \varepsilon_i > 0 \) (i \( \in \) I) . Assume that \( q \) is a projector from \( \mathcal{A}' \) such that \( 0 \leq x \leq q \) implies \( (a_i x\xi, \xi) < \varepsilon_i (x\xi, \xi) \) (i \( \in \) I) . Then there exists some \( p \in \text{Proj} \mathcal{A} \) such that \( \phi(p) \geq (q\xi, \xi) \) and \( \|p a_i p\|_\infty \leq \varepsilon_i \) for i \( \in \) I.

(F) Let \( 0 \leq x \leq q \) and \( 0 \leq r \leq q' \) , such that \( (xy\xi, \xi) \leq (y\xi, \xi) \) for all \( 0 \leq y \leq r \) , \( y \in \mathcal{A}' \). If \( q = \text{support} x \) , then, for each \( 0 \leq y \leq q \), the inequality \( (xy\xi, \xi) \leq (y\xi, \xi) \) holds.

(G) (Maximal ergodic lemma). Let \( 0 \leq a_i \leq q \) (i = 1, 2, 3, 4) , \( \varepsilon > 0 \) , be given and let \( \phi(a_i^2) < \varepsilon^3 \). Then there is a projector \( p \in \mathcal{A} \)
with \( \phi(p) \geq 1 - 8\epsilon \) and such that \( \|s_n(\alpha_i)p\|_\infty < \epsilon \) for all \( i = 1, 2, 3, 4 \) and \( n = 1, 2, \ldots \).

(H) Let \( \epsilon > 0 \), \( x \in \mathcal{A} \). Then there exist a positive integer \( N \) and \( p \in \text{Proj } \mathcal{A} \) with \( \phi(p^4) < \epsilon \), such that, for \( n > N \),

\[
\|s_n(x) - \hat{x}\|p\|_\infty < \epsilon,
\]

holds, where \( \hat{x} \) is defined as in (A).

(I) The conditions in (H) imply the almost uniform convergence of \( s_n(x) \) to \( \hat{x} \).

A few comments are necessary. (A) and (B) are rather easy modifications of the well-known classical results (see for. ex. [28]). (D), (E), (F) are slight simplifications of Goldsteins lemmas [12]. It should be noticed here that Goldstein does not use the facts expressed in (B), (C), (H), (I), and that is why his proof needs some other considerations. The simple proof of (E) is due to Luczak [23]. In fact, we do not use (E) in all its generality but the proof of a lemma we need should be almost the same. (I) was first noticed by A. Paszkiewicz [25]. In general, the proof presented here should be treated as the simplification of Goldstein's ideas. For the sake of completeness, we prove shortly all the steps.

(A) It is enough to show (A) for \( x \in \mathcal{A}_s \). By the well-known Kadison inequality [16] and the properties of \( \alpha \), we have

\( \phi((\alpha x)^2) \leq \phi(x^2) \).

This means that \( \alpha \) (after its unique extension) is a contraction in \( L_2(\mathcal{A}_s, \phi) \), so \( 1/n \sum_{k=0}^{n-1} \alpha^k + e \) strongly in \( L_2(\mathcal{A}, \phi) \), where \( e \) is a suitable projection. Hence, for any hermitian \( x \in \mathcal{A} \) and \( y \in \mathcal{A}' \), we have \( s_n(x)(y) = y \sum_{k=0}^{n-1} \alpha^k x \xi + y e(x\xi) \) in the norm of \( L_2(\mathcal{A}_s, \phi) \) (hence in the norm of \( \mathcal{R}_\phi \)). Since the sequence \( s_n(x) \) is uniformly bounded and the set \( \{y\xi, y \in \mathcal{A}'\} \) is dense in \( \mathcal{R}_\phi \), \( s_n(x) \) strongly, and \( x \in \mathcal{A} \) (because \( \mathcal{A} \) is strongly closed). The \( \alpha \)-invariance of \( \hat{x} \) is obvious.

(B) Let \( x \in L_2(\mathcal{A}_s, \phi) \). Then \( (x - \alpha x, y) = 0 \) for all \( y \in \mathcal{A}_s \). Consequently, since \( \alpha \) is a contraction, \( x = \alpha x \) [28]. Let \( x_j \in \mathcal{A}_s \), \( x_j + x \) in \( L_2(\mathcal{A}_s, \phi) \). We have \( s_n(x_j) + \hat{x}_j \) in \( L_2 \) by (A). For \( \epsilon > 0 \), we find \( k_0 \) and \( n_0 \), such that \( \|x - x_{k_0}\|_2 < \epsilon/2 \) and \( \|s_{n_0}(x_{k_0}) - \hat{x}_{k_0}\|_2 < \epsilon/2 \). Of course, here \( \|x\|_2 = \phi(x^2) \). Since \( s_{n_0}(x) \equiv x \), we obtain

\[
\|x - \hat{x}_{k_0}\|_2 \leq \|s_{n_0}(x) - s_{n_0}(x_{k_0})\|_2 + \|s_{n_0}(x_{k_0}) - \hat{x}_{k_0}\|_2 < \epsilon \text{ for } k_0 \text{ large enough.}
\]
Taking into account the fact that all $\hat{x}_k$'s are $\alpha$-invariant, we obtain the formula we are looking for.

(C) Obviously, it is enough to show that for $x \in \mathcal{A}_s$ and $\varepsilon > 0$ there exist elements $a_1, a_2 \in \mathcal{A}_+$ with $\phi(a_1^2) < \varepsilon^2$ and the elements $y_1, y_2, \ldots, y_k \in \mathcal{A}$ such that

\[
(*) \quad x - \hat{x} = a_1 - a_2 + \sum_{i=1}^{k} (y_i - \alpha y_i).
\]

By (B), there exist elements $b, h, y_1, \ldots, y_k$ in $\mathcal{A}_s$ such that

\[
x - x = b + h + \sum_{i=1}^{k} (y_i - \alpha y_i)
\]

holds, with $\|b\|_2 < \varepsilon/2$, $\alpha h = h$. We then have $\|s_n(x) - \hat{x} - h\|_2 \leq \|s_n(b)\|_2 + \|s_n(\sum_{i=1}^{k} (y_i - \alpha_i))\|_2$. Passing to the limit with $n \to \infty$, we obtain $\|h\|_2 \leq \lim \sup \|s_n(b)\|_2 < \varepsilon^2/2$ (since $\alpha$ is a contraction in $L_2(\mathcal{A}_s, \phi)$). Consequently, we have $\|h\|_2 < \varepsilon/2$.

Putting $a = b + h$, and then $a = a_1 - a_2$ with $a_i \geq 0$ we obtain the equality $(*)$ with $\phi(a_1^2) < \varepsilon^2$.

(D) It is a simple consequence of Dixmier's lemma formulated in (a). In fact, it is enough to put (for some fixed $0 \leq y \in \mathcal{A}'$)

\[
\psi(x) = (\alpha(x) y \xi, \xi, \xi). \quad \text{Then, by Dixmier's lemma there exists } z \in \mathcal{A}' \text{ such that } \psi(x) = (x z \xi, z \xi) = (x|z|^2 \xi, \xi). \quad \text{Put } |z|^2 = \alpha'(y). \quad \text{The standard reasoning completes the proof of (D). Let us notice here that the proof of Dixmier's lemma is simple and short ([6], p. 50).}

(E) It is enough to choose $p$ as the projection onto the subspace $\mathcal{X}'_{\alpha} = [\rho q; \rho \in \mathcal{A}']$. In fact, we then have $p \in \mathcal{A}$ (because $\mathcal{A}'_{\alpha} = X$). Moreover, for $\rho \in \mathcal{A}'$, we have $(p_\alpha \pi q \rho q \xi) = (\alpha_1 p q \rho q \xi) = (\alpha_1 q \rho q \xi, q \rho q \xi) = (\alpha_1 (q^* q \rho q \xi), \xi, \xi) \leq \varepsilon_1 (q^* q \rho q \xi, \xi, \xi) = \varepsilon_1 (q \rho q \xi, \xi)$. Thus $p_\alpha p \leq \varepsilon_1$ over $[\rho q \xi, \rho \in \mathcal{A}']$. On the subspace $\mathcal{X}_{\alpha}$, $p_\alpha p = 0$. Consequently, $p_\alpha p \leq \varepsilon_1$ and $\|p_\alpha p\| \leq \varepsilon_1$ follows. Evidently, $\phi(p) = (p \xi, \xi) \geq (q \xi, \xi)$.

(F) The proof is very standard. If $0 \leq y \leq cr$ for some $c > 0$ , $(y \in \mathcal{A}'_q)$, then $0 \leq c^{-1} y \leq r$ and also $(x y \xi, \xi) \leq (y \xi, \xi)$. Let $0 \leq y \in \mathcal{A}'q$ and $q_n = \text{the value of the spectral measure of } r \text{ on the interval } (1/n, \infty)$. Then $q_n \leq nr$ and $q_n \to q$ (strongly). Also $y \leq \|y\|_{\infty} q$, and $q_n y q_n \leq \|y\|_{\infty} q_n$. The passing to the limit in the inequality $(x q_n y q_n \xi, \xi) \leq (q_n y q_n \xi, \xi)$ ends the proof.

(G) This is the key point of the proof and it differs from Goldstein's ([12], Theorem 1.2) only in some details. Let

\[
L = \{y = (y_{ik})_{i=1,k=1,2,\ldots,N}^i \leq 1, 0 \leq y_{ik} \in \mathcal{A}'_q, \sum_{i,k} y_{ik} \leq 1\}.
\]

For $y \in L$, we put
and \( g(y) = \frac{4}{k} g_i(y) \). By the weak compactness of \( L \), there is a matrix \( \bar{y} = (\bar{y}_{ik}) \in L \) such that \( g(\bar{y}) = \max \limits_L g \). Let \( x_N = 1 - \frac{4}{k} \sum \limits_{i=1}^{N} \bar{y}_{ik} \) and let \( 0 \leq x \leq x_N \), \( x \in \mathcal{A}' \). For \( 1 \leq n \leq 4 \), we put \( y_{ij} = \bar{y}_{ij} \) if \( (i, j) \neq (n, k) \), and \( y_{nk} = \bar{y}_{nk} + x \). Then such \( y \) belongs to \( L \) and, consequently, \( g(y) \leq g(\bar{y}) \) and \( (s_k(a_n^2)x, \xi) \leq (x, \xi) \). Let \( q_{N,2} = \text{support} \ x_N \). Then, by (F), for \( 0 \leq x \in q_{N,2} \) we also have \( (s_k(a_n^2)x, \xi) \leq (x, \xi) \). By (E), there is a projector \( p_N \in \mathcal{A}' \) such that \( \phi(p_N) \geq (q_{N,2}, \xi) \) and \( \| p_N s_k(a_n^2) - p_N \| < \varepsilon^2 \) for \( n = 1, 2, 3, 4 \) and \( k = 1, 2, \ldots, N \). Put \( y_{11} = a'(y_{12}), \ldots, y_{1,N-1} = a'(y_{1,N}), \) \( y_{i,N} = 0 \) for \( i = 1, 2, 3, 4 \), where \( a' \) is defined as in (D). Then \( y = (y_{ik}) \in L \) and \( g(y) \leq g(\bar{y}) \). Moreover, by the properties of \( a' \), after simple calculations we obtain (for \( n = 1, 2, 3, 4 \))

\[
g_n(y) = \sum \limits_{k=1}^{N} k(\bar{y}_{nk}, \xi) + g_n(y) + \sum \limits_{k=1}^{N} k(y_{nk}, \xi) - \sum \limits_{k=1}^{N} k(\bar{y}_{nk}, \xi)
\]

and

\[
\sum \limits_{k=1}^{N} k(y_{nk}, \xi) = \sum \limits_{k=1}^{N-1} k(a'(y_{n,k+1}), \xi) \leq \sum \limits_{k=2}^{N} (k - 1)(\bar{y}_{nk}, \xi).
\]

From the last two inequalities we obtain

\[
g_n(y) \leq g_n(y) + \sum \limits_{k=1}^{N} (\bar{y}_{nk}, \xi) - \sum \limits_{k=1}^{N} (\bar{y}_{nk}, \xi)
\]

and, consequently, since \( g(\bar{y}) \leq g(y) \), therefore

\[
\sum \limits_{n=1}^{N} \sum \limits_{k=1}^{N} (\bar{y}_{nk}, \xi) \leq \sum \limits_{n=1}^{N} (\bar{y}_{nk}, \xi) = \sum \limits_{n=1}^{N} \phi(a_n^2) < 4\varepsilon.
\]

The inequality \( \sum \limits_{k=1}^{N} \bar{y}_{nk} \leq 1 \) gives

\[
\sum \limits_{n=1}^{N} \sum \limits_{k=1}^{N} (\bar{y}_{nk}, \xi) \leq \sum \limits_{n=1}^{N} (\bar{y}_{nk}, \xi) = \sum \limits_{n=1}^{N} \phi(a_n^2) < 4\varepsilon.
\]

Since \( 0 \leq x_N \leq 1 \), we have \( (q_{N,2}, \xi) \geq (x_N, \xi) \geq 1 - 4\varepsilon \). Choose \( N_s \) in such a way that \( p_{N_s} \) converge weakly to some operator

\[
Q = \int \lambda d\mu(d\lambda) \quad \text{(spectral representation)}.
\]

Put \( p = 1 - E[0, 1/2) \).
Since $1 - Q \geq 1/2(1 - p)$, by a standard reasoning, we obtain

$$\phi(p) \geq 1 - 8\varepsilon.$$ 

Moreover, we have, by the Kadison inequality,

$$\|s_k(a_n)p_n\|_\infty = \|p_n^*s_k(a_n)^2p_n\|_\infty \leq \|p_n^*s_k(a_n^2)p_n\|_\infty < \varepsilon^2.$$ 

Thus

$$\|s_k(a_n)p\|_s \leq \lim_{s} \|s_k(a_n)p_n\|_\infty < \varepsilon$$

for all $k = 1, 2, \ldots$ and $n = 1, 2, 3, 4$.

(H) Follows easily in a standard way from (A), (C) and (G).

(I) Evidently, it is enough to prove the following general fact.

Let $\{x_n\}$ be a bounded sequence of elements of $\mathcal{A}$ (say $\|x_n\|_\infty \leq 1$), satisfying the condition

(*) for each $\varepsilon > 0$, there are some $p \in \text{Proj} \mathcal{A}$ and $N$, such that

$$\phi(p) \geq 1 - \varepsilon$$

and $\|x_n p\|_\infty < \varepsilon$ for $n > N$.

Then $x_n \rightarrow 0$ almost uniformly.

We first prove that (*) implies

(**) for each $\varepsilon > 0$ and for each $q \in \text{Proj} \mathcal{A}$, there is an $r \in \text{Proj} \mathcal{A}$ such that $r \leq q$, $\phi(q - r) < \varepsilon$ and $\|x_n r\|_\infty < \varepsilon$ for $n$ large enough.

This implication follows easily from the facts (a) and (b) indicated at the beginning of this section. Indeed, let $0 < \varepsilon_n \rightarrow 0$.

By (*), we can find a sequence $\{r_n\} \subset \text{Proj} \mathcal{A}$ with $\phi(r_n^+) < \varepsilon_n$ and a sequence of positive integers $m(n)$, such that $\|x_m r_n\|_\infty < \varepsilon_n$ for $m > m(n)$. Let $q \in \text{Proj} \mathcal{A}$ be given. Then $\phi(q r_n^+) \rightarrow 0$ and we can fix $n_0$ such that $\varepsilon_n < \varepsilon$ and $\phi(q r_n^+) < \varepsilon^4$. Putting

$r = q e^{q r_n^+} \big| 0, e^2 \big)$, we have $r \leq q$, $\phi(q - r) < \varepsilon$ and $\|r_n^+ r\|_\infty < \varepsilon$.

Moreover, $\|x_m r_n\|_\infty < \varepsilon_n < \varepsilon$ for $m > m(n_0)$ (comp. (y)). By (b), we then have $\|x_m r\| < 2\varepsilon$ for $m > m(n_0)$.

Let us fix some $\varepsilon > 0$. By (**), we find a sequence $\{p_n\} \subset \text{Proj} \mathcal{A}$ such that $1 \geq p_1 \geq p_2, \ldots$, $\phi(p_n - p_{n+1}) < 2^{-n} \varepsilon$ and $\|x_m p_n\|_\infty < \varepsilon$ for $m > m(n)$. Put $p = \inf_k p_k$. Then $\phi(p) = \sum_{n} \phi(p_n - p_{n+1}) < \varepsilon$. Moreover, $\|x_m p\|_\infty < \varepsilon$ for $m > m(n_0)$. This means that $x_m \rightarrow 0$ almost uniformly.

2. In this section we are going to discuss briefly some subadditive limit theorems. It is well-known that, for each sequence $\{c_n\}$ of real numbers satisfying the condition

$$c_{n+k} \leq c_n + c_k$$

(for all positive integers $n, k$),

we have $\frac{1}{n} c_n + \inf_k \frac{1}{k} c_k$ as $n \rightarrow \infty$. This simple fact contains, to some extent, the basic idea of a much deeper result of Kingman.
which can be formulated as follows:

THEOREM OF KINGMAN [18] [19]. Let \((\Omega, B, p)\) be a probability space, and let \(\theta\) be a measure preserving transformation on \(\Omega\). Let \(\{f_n\}\) be a sequence of integrable functions on \((\Omega, B, p)\), such that

\[
f_{n+k} \leq f_n + f_k \circ \theta^n \quad \text{for all } n, k,
\]

and let \(\inf_n f_n^\infty /n \, dp > -\infty\). Then the sequence \(1/n f_n\) converges in \(L_1\) and almost everywhere.

One can generalize this result to the von Neumann algebra context in the following way. Let \(\mathcal{A}\) be a von Neumann algebra with a finite faithful normal trace \(\tau\). \(L_1(\mathcal{A}, \tau)\) stands for the space of Segal (of operators integrable with respect to \(\tau\)); see for ex. [30] [24] [35]). Let \(\|\cdot\|_1\) denote the norm in \(L_1(\mathcal{A}, \tau)\), i.e. \(\|x\|_1 = \tau(|x|)\) where \(|x| = (x^*x)^{1/2}\). We would like to formulate here one of the results which can be treated as a non-commutative analogue of Kingman. We shall need one definition. A sequence \(\{\xi_n\}\) of elements in \(L_1(\mathcal{A}, \tau)\) (unbounded in general) is said to be subadditive if there is a \(\tau\)-preserving \(*\)-automorphism \(\alpha\) of \(\mathcal{A}\) such that, for all positive integers \(n, k\), we have

\[
(*) \quad \xi_{n+k} \leq \xi_n + \alpha^n \xi_k.
\]

In the case of equality in \((*)\), we have

\[
(2) \quad \xi_n = \sum_{k=0}^{n-1} \alpha^k \xi_1.
\]

so then \(\{\xi_n\}\) is a sequence of ergodic averages. We need one more definition. A sequence \(\{x_n\} \subset L_1(\mathcal{A}, \tau)\) is said to be weakly almost uniformly convergent to an element \(x \in L_1(\mathcal{A}, \tau)\) if for each \(\varepsilon > 0\) there exists some \(p \in \text{Proj } \mathcal{A}\) with \(\tau(p^\perp) < \varepsilon\) and such that \((x_n - x)p \in \mathcal{A}\) for a sufficiently large end \(\|p(x_n - x)p\|_\infty \to 0\).

2.1 THEOREM [14]. If \(\xi_n; (n = 0, 1, 2, \ldots)\) is a subadditive sequence in \(L_1(\mathcal{A}, \tau)\) such that \(\inf_n n^{-1} \tau(\xi_n) > -\infty\) then \(n^{-1} \xi_n\) converges in \(L_1\)-norm and weakly almost uniformly to an \(\alpha\)-invariant element \(\xi \in L_1\). For the proof, we refer the reader to [14].

3. Let \(\mathcal{A}\) be a von Neumann algebra with a faithful normal state \(\phi\). In this section we assume that \(\mathcal{A}\) has the separable predual \(\mathcal{A}_*\). This is equivalent to the assumption that \(\mathcal{A}\) acts in a separable Hilbert space.

Let \((T, F, m)\) be a probability space. Denote by \((\Omega, B, p)\) the direct product of a countable sequence of copies of \((T, F, m)\). In particular, \(\Omega = \{\omega: (t_1, t_2, \ldots); t_j \in T\}\). The product measure.
\[ p = m \otimes m \otimes \ldots \] will also be denoted by \[ dp = dt_1 \otimes dt_2 \ldots \]. Let us denote by \( \mathcal{G}(\mathcal{A}, \phi) \) the class of all normal \(*\)-endomorphisms \( \alpha \) of \( \mathcal{A} \), such that \( \phi \) is \( \alpha \)-invariant and \( \alpha 1 = 1 \). A function \( f: T \rightarrow \mathcal{A} \) is said to be ultraweakly \( m \)-measurable if, for each \( \alpha_x \in \mathcal{A}_x^* \), the function \( t \rightarrow \langle f(t), \alpha_x \rangle \) is \( m \)-measurable. Very recently, N. Dang Ngoc [5] has proved the following

**RANDOM ERGODIC THEOREM IN VON NEUMANN ALGEBRAS.** Let \( \xi: T \rightarrow \mathcal{G}(\mathcal{A}, \phi) \) be a map such that, for each \( x \in \mathcal{A} \), the function \( t \rightarrow \xi(t)x \) is ultraweakly \( m \)-measurable. Then, for each \( x \in \mathcal{A} \), there exists \( \hat{x} \in \mathcal{A} \) such that, for \( p \)-almost every \( \omega = (t_1, t_2, \ldots) \), we have that

\[
S_n(x, \omega) = n^{-1} \sum_{k=1}^{n} \xi(t_k)x + \hat{x}
\]

almost uniformly in \( \mathcal{A} \). Moreover, \( \hat{x} \) does not depend on \( \omega \) and \( \hat{x} \) is \( \xi(t) \)-invariant for \( m \)-almost every \( t \in T \).

The result just formulated is the non-commutative version of the Kakutani-Ryll Nardzewski random ergodic theorem (comp. [17], [27]). For the sake of completeness, we sketch the proof. The main idea of the proof presented here is the same as in [4] but our proof differs in its key point (Proposition 3) from the proof of Dang Ngoc. Namely, Dang Ngoc follows the method indicated by Ryll Nardzewski [27], and we follow the idea of Gladysz [11] who uses the Andersen-Jessen theorem. It leads us to a rather interesting application of vector-valued martingales.

**Sketch of the proof.** Let \( M = L_1(T, F, m) \). Consider the \( W^* \)-tensor product algebra \( B = \mathcal{A} \otimes M \). Then \( B \) can be identified with the von Neumann algebra \( L_1(T, \tau, \mathcal{A}) \) of essentially bounded ultraweakly \( m \)-measurable functions \( f: T \rightarrow \mathcal{A} \) with the norm \( \|f\|_\infty = \sup_{t \in T} \|f(t)\|_\mathcal{A} \) ([29], p. 68).

The tensor product state \( \nu = \phi \otimes m \) on \( B \) is given by the formula

\[
\nu(f) = \int \phi(f(t)) \, m(dt) \quad \text{for } f \in B.
\]

Let \( Q \) be a measure \( m \)-preserving transformation of \( T \), and let \( \xi: T \rightarrow \mathcal{G}(\mathcal{A}, \phi) \) be a map such that \( t \rightarrow \xi(t)x \) is ultraweakly \( m \)-measurable for each \( x \in \mathcal{A} \). Then we have the following

**Proposition 1.** For each \( x \in \mathcal{A} \), there is a ultraweakly \( m \)-measurable map \( \hat{x}: T \rightarrow \mathcal{A} \) such that, for each \( d > 0 \),
\[ \sup \operatorname{ess} \left\| \left( n^{-1} \sum_{k=1}^{n} \xi(t) \xi(Q_t) \ldots \xi(Q^{k}t)x - x \right) J_d(t) \right\| \to 0 \]

where \( J_d : T \to \operatorname{Proj} A \) is a suitably chosen ultraweakly \( m \)-measurable map satisfying the inequality

\[ \int_T \phi(J_d(t)) \, m(dt) \geq 1 - d. \]

The proof of Proposition 1 can easily be obtained by applying Theorem of Kümmner to the mapping \( \Phi \) defined by the formula

\[ \Phi(f)(t) = \xi(t) f(Q_t). \]

Proposition 2. Under the assumptions of Proposition 1, for each \( x \in A \), there exists an ultraweakly \( m \)-measurable map \( \hat{x} : T \to A \) such that, for \( m \)-almost all \( t \), the averages \( s_n(t, x) \) converge almost uniformly in \( A \) to \( \hat{x}(t) \).

We omit the rather standard proof.

Let us now denote by \( \tau \) the shift transformation in \( \Omega \), i.e. \( \tau((t_1, t_2, \ldots)) = (t_2, t_3, \ldots) \). Of course, \( \tau \) preserves the measure \( p = m \otimes m \otimes \ldots \).

Put \( \eta(\omega) = \xi(t_1) \) for \( \omega = (t_1, t_2, \ldots) \), and let

\[ D = A \otimes L_\infty(\Omega, B, p) = L_\infty(\Omega, B, A). \]

Put

\[ (qg)(\omega) = \eta(\omega) g(\tau \omega) \]

and

\[ \gamma(g) = \int_\Omega \phi(g(\omega)) \, p(d\omega). \]

It is easy to verify that \( g \in G(D, \gamma) \). By Propositions 1 and 2, there exists an element \( \hat{x} \in D \) such that, for \( p \)-almost all \( \omega \in \Omega \),

\[ s_n(x, \omega) \to \hat{x}(\omega) \]

almost uniformly.

It suffices to show that \( \hat{x} \) does not depend on \( \omega \). This follows from the following

Proposition 3. Let \( \hat{x} \in D \). If \( \hat{x} \) is \( q \)-invariant, then \( \hat{x} \equiv \operatorname{const}(\mathcal{E}^A) \) \( p \)-almost everywhere.

Proof. Put

\[ z = \int \hat{x}(t_1, t_2, \ldots) \, dt_1 dt_2 \ldots \]
(the integral being taken in the weak*-sense).

\[ qz = \xi(t_1) \int \hat{x}(t_2, t_3, \ldots) \, dt_2 \, dt_3 \ldots \]

\[ = \int \xi(t_1) \hat{x}(t_2, t_3, \ldots) \, dt_2 \, dt_3 \ldots \]

\[ = \int q\hat{x}((t_1, t_2, \ldots)) \, dt_2 \, dt_3 \ldots \]

\[ = \int \hat{x}(t_1, t_2, \ldots) \, dt_2 \, dt_3 \ldots . \]

Similarly, we obtain

\[ q^nz = \int \hat{x}(t_1, t_2, \ldots) \, dt_{n+1} \, dt_{n+2} \ldots \]

for \( n = 1, 2, \ldots . \)

Let \( \mathcal{A} \) act in a Hilbert space \( \mathcal{H} \). Since the predual \( \mathcal{A}^*_\mathcal{H} \) of \( \mathcal{A} \)

is separable, the Hilbert space \( \mathcal{H} \) can be assumed separable, too. Let

us remark that, for each \( x \in \mathcal{H} \) and \( h \in \mathcal{H} \), the function

\[ x_h: \omega \mapsto x(\omega)h \]

is Bochner \( p \)-integrable. Indeed, \( x_h \) is separably-valued and weakly \( m \)-measurable (because \( x + x(t) \) is weak*-\( m \)-measurable). Thus \( x_h \) is strongly measurable. Moreover,

\[ \int \|x(\omega)\|^p \, d\omega \leq \sup \text{ess} \|x(\omega)\| \cdot \|h\| < \infty. \]

Consequently, \( x_h \) is Bochner \( p \)-integrable (comp. [13], Theorem 3.53), and the sequence

\[ q^nz = \int \hat{x}(t_1, t_2, \ldots) \, dt_{n+1} \, dt_{n+2} \ldots \]

\( (n = 1, 2, \ldots) \)

can be treated as the \( \mathcal{H} \)-valued martingale

\[ q^nz = E(x_h|Y_n) \]

\( (n = 1, 2, \ldots) \)

where \( E(\cdot|Y_n) \) denotes the conditional expectation with respect to

the \( \sigma \)-field \( Y_n \) of measurability of the first \( n \) coordinates:

\[ t_1 = t_1(\omega), \ t_2 = t_2(\omega), \ldots, \ t_n = t_n(\omega), \ \omega \in \Omega. \]

(Compare [2], [3], [7]). By the vector martingale convergence theorem (Chatterji [3]),

for each \( h \in \mathcal{H} \), there is a set \( \Omega_h \subset \Omega \) with \( p(\Omega_h) = 0 \), such that

\[ (q^nz)(\omega)h \to \hat{x}(\omega)h \quad \text{for } \omega \in \Omega - \Omega_h. \]

Let \( h_1, h_2, \ldots \) be a sequence dense in \( \mathcal{H} \). Put

\[ \Omega_0 = \bigcup_{i=1}^{\infty} \Omega_{h_i} \]

where \( \Omega_{h_i} \) is the set with \( p(\Omega_i) = 0 \) for which

\[ \sup_{\omega \in \Omega_{h_i}} \|\hat{x}(\omega)\| = \sup_{\omega \in \Omega} \text{ess} \|\hat{x}(\omega)\| = M < \infty. \]

Then \( p(\Omega_0) = 0 \) and

\[ (q^nz - \hat{x})(\omega)h \to 0 \]

for each \( h \in \mathcal{H} \) and \( \omega \in \Omega - \Omega_0 \). This follows easily from the fact

that \( q^nz = q^nz - \hat{x} \), and that, for \( \omega \in \Omega_1 \), we have
Taking account of formula (4) and

\[ |q^n y|^2 = q^n |y|^2 \quad \text{where} \quad |y|^2 = y \star y \]

(q is an endomorphism of D), we obtain

\[ (q^n |z - \hat{x}|^2(\omega)h, h) \to 0 \]

for \( h \in \mathcal{H} \) and \( \omega \in \Omega - \Omega_0 \) and, consequently,

\[ \phi(q^n |z - \hat{x}|^2(\omega)) \to 0 \quad \text{for almost all} \ \omega \]

Finally, since the state \( \gamma \) is \( q \)-invariant,

\[ \gamma(|z - \hat{x}|^2) = \gamma(q^n |z - \hat{x}|^2) \to 0 \quad \text{as} \ \ n \to \infty, \]

which implies \( \hat{x}(\omega) = z \) \( p \)-almost everywhere. The proof is completed.

FINAL REMARKS. One of the deepest and most beautiful results in the classical ergodic theory has been obtained recently by M.A. Akcoglu [1] on the positive contraction in \( L_p \). Also, there are very interesting results for not necessarily positive contractions, see for ex. R. Duncan [8]. It would be very interesting to prove some analogues of the theorems just mentioned. It seems to be very probable that some new methods will be needed.

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A REMARK ON THE CENTRAL LIMIT THEOREM IN BANACH SPACES

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It is known that in cotype 2 or type 2 Banach spaces, pregaussian bounded random variables satisfy the central limit theorem (CLT). Conversely, by an example of S.A. Chobanyan and V.I. Tarieladze [1], if in a Banach space E, every pregaussian bounded random variable satisfies the CLT, necessarily E is of finite cotype; nothing more can be said about the cotype and type properties of E since for example the spaces $\ell_p(E)$, where E is a Banach space of cotype 2 and $1 \leq p < \infty$, behave in regard to the classical CLT like the usual $\ell_p$ spaces (cf. [5]). Our aim in this paper is to construct a Banach space of cotype $2 + \delta$ and type $2 - \delta$ for every $\delta > 0$ in which there exists a pregaussian bounded random variable which does not satisfy the CLT. The starting point to this note is the paper [2] by E. Giné and J. Zinn where an example in the same spirit is construct in $\ell_2(E)$ when E is not of cotype 2.

Let E be a real separable Banach space. If X is an E-valued random variable, $(X_n)_{n \in \mathbb{N}}$ will denote a sequence of independent copies of X and $S_n(X)$ the partial sum $X_1 + \ldots + X_n$. With these notations, X is said to satisfy the central limit theorem (CLT) if the sequence $(S_n(X)/\sqrt{n})_{n \in \mathbb{N}}$ converges in law. It is known that X is then necessarily pregaussian (i.e. X is centered and there exists an E-valued Gaussian random variable with the same covariance structure as X) and $\lim_{t \to \infty} t^2 P\left\{ \|X\| > t \right\} = 0$. 

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In this paper we will be mainly concerned with Banach spaces $E$ of the form $l^p((B_k)_{k \in \mathbb{N}})$ where $1 \leq p < \infty$ and $(B_k)_{k \in \mathbb{N}}$ is a sequence of real separable Banach spaces; $l^p((B_k)_{k \in \mathbb{N}})$ denotes the Banach space consisting of all sequences $x = (x_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} B_k$ with $\|x\| = \left( \sum_{k \in \mathbb{N}} \|x_k\|^p \right)^{1/p} < \infty$. When all the $B_k$ are identical, $l^p((B_k)_{k \in \mathbb{N}})$ will be denoted by $l_p(B)$ where $B = B_k$.

The following theorem proved by E. Giné and J. Zinn [2] (see also [7] as well as [5] for the case $1 \leq p < 2$) provided the first examples of Banach spaces of type 2 in which the CLT under the classical necessary conditions fails.

**Theorem.** Let $1 \leq p \leq 2$ and $B$ be a real separable Banach space. If $B$ is not of cotype 2, there exists in $l_p(B)$ a pregaussian random variable $X$ such that $\lim_{t \to \infty} t^2 P[\|X\| > t] = 0$ which does not satisfy the CLT.

One should notice that the proof of this theorem is based on an additional necessary condition for the CLT in spaces of the form $l^p((B_k)_{k \in \mathbb{N}})$ which we state here as a lemma.

**Lemma.** Let $X = (X_k)_{k \in \mathbb{N}}$ be a random variable taking its values in $l^p((B_k)_{k \in \mathbb{N}})$ satisfying the CLT. Then:

$$\lim_{n \to \infty} \sum_{k \in \mathbb{N}} E \left[ \max_{1 \leq j \leq n} \left( \frac{\|x_k\|}{\sqrt{n}} \right)^p 1_{\left[ \|x_k\| \leq \sqrt{n} \right]} \right] = 0.$$  

**Proof.** We first assume that $X$ is symmetrically distributed. Then, by Lévy's and Hoffmann-Jørgensen's inequalities [3], for every $n$:

$$\sum_{k \in \mathbb{N}} E \left[ \max_{1 \leq j \leq n} \left( \frac{\|x_k\|}{\sqrt{n}} \right)^p 1_{\left[ \|x_k\| \leq \sqrt{n} \right]} \right]$$
\[ \begin{aligned}
&\leq 2 \mathbb{E}\left[ \left( \sum_{j=1}^{n} \frac{X_j}{\sqrt{n}} \right)^p \mathbb{I}\{ \|X\| \leq \sqrt{n} \} \right]^p \\
&\leq C \left( \mathbb{E}\left[ \max_{1 \leq j \leq n} \left( \frac{X_j}{\sqrt{n}} \right)^p \right] + \mathbb{E}\left[ \left( \frac{S_n(X)}{\sqrt{n}} \right)^p \right] \right) \\
&\leq C \left( \int_0^1 p\left( \frac{S_n(X)}{\sqrt{n}} > t\sqrt{n} \right) dt^p + \mathbb{E}\left[ \left( \frac{S_n(X)}{\sqrt{n}} \right)^p \right] \right) \\
&\leq C \left( \mathbb{E}\left[ \frac{S_n(X)}{\sqrt{n}} \right]^p + \mathbb{E}\left[ \frac{S_n(X)}{\sqrt{n}} \right]^p \right)
\end{aligned} \]

where $C$ is a constant depending only on $p$ which may change from line to line.

This inequality still holds if $X$ is only centered by simply replacing $X$ by $\varepsilon X$ where $\varepsilon$ is a Rademacher random variable independent of $X$. Now, if $X$ verifies the CLT, by the usual approximation argument [6], there exists, for every $\delta > 0$, a simple mean zero random variable $Y$ such that:

\[ \sup_{n \in \mathbb{N}} \mathbb{E}\left[ \frac{\|S_n(X - Y)\|}{\sqrt{n}} \right] < \delta. \]

Applying the above inequality to $X - Y$ and letting $n$ tend to infinity proves the lemma.

**Remark.** If $p > 2$ (and only in this case), (1) is a direct consequence of the classical necessary condition $\lim_{t \to \infty} t^2 \mathbb{P}\{ \|X\| > t \} = 0$. Indeed:

\[ \begin{aligned}
&\sum_{k \in \mathbb{N}} \mathbb{E}\left[ \max_{1 \leq j \leq n} \left( \frac{X_j}{\sqrt{n}} \right)^p \mathbb{I}\{ \|X\| \leq \sqrt{n} \} \right] \\
&\leq n \mathbb{E}\left[ \left( \frac{\|X\|}{\sqrt{n}} \right)^p \mathbb{I}\{ \|X\| \leq \sqrt{n} \} \right] \\
&\leq \int_0^1 nt^2 \mathbb{P}\{ \|X\| > t\sqrt{n} \} \frac{dt^p}{t^2}
\end{aligned} \]

and the conclusion follows by dominated convergence.
A consequence of the preceding theorem is the existence of a Banach space of type 2 and cotype q for every q > 2 in which one can find a pregaussian random variable X such that
\[ \lim_{t \to \infty} t^2 \mathbb{P}(\|X\| > t) = 0 \]
which fails to satisfy the CLT. (An explicit Banach space can be obtained by taking
\[ E = \ell_2(\{ \frac{n_k}{p_k} \}_{k \in \mathbb{N}}) \]
for some appropriate sequences \( \{n_k\}_{k \in \mathbb{N}} \) and \( \{p_k\}_{k \in \mathbb{N}} \) of respectively strictly increasing integers and strictly decreasing real numbers converging to 2 using the arguments of E. Giné and J. Zinn with \( B = \ell_p, p > 2 \), and finite dimensional approximations.) Thus, even in type 2 spaces with good cotype properties, the CLT under the classical necessary conditions is not satisfied. However, in type 2 spaces, one can still obtain information on the CLT since, by the well-known result of J. Hoffmann-Jørjensen and G. Pisier [4], every mean zero strongly square integrable random variable taking its values in a type 2 space satisfies the CLT. But, as for the cotype 2, only a slightly weakening of the type 2 hypothesis can lead to worse CLT's. For example, by an easy change of probability in J. Hoffmann-Jørjensen and G. Pisier's result, we see that there exist Banach spaces of cotype 2 and type p for every p < 2 in which bounded random variables do not necessarily verify the CLT.

Our example will take part of the two precedings: we will construct a Banach space of cotype 2 + δ and type 2 − δ for every δ > 0 in which there exist bounded pregaussian random variable which do not satisfy the CLT. S.A. Chobanyan and V.I. Tarieladze [1] had already observed that such a situation appears in every Banach space in which \( c_0 \) is finitely representable (that is of no finite cotype). This example shows that this is possible even in spaces with good cotype and type properties.

The construction we present is directly inspired from the proof of the theorem of E. Giné and J. Zinn. Take B to be a Banach space of type 2 and cotype q for every q > 2 but not of cotype 2 (*). Since in cotype 2 spaces every

(*): See the final remark.
pregaussian random variable is square integrable in norm, for every \( c \), \( 0 < c < \infty \), there exists a symmetric pregaussian random variable \( Y \) taking values in \( B \) with associated Gaussian random variable \( G(Y) \) such that:

\[
E[\|Y\|^2] = 1 \quad \text{and} \quad E[\|G(Y)\|^2] < c.
\]

After a change of probability we can even suppose that:

\[
\|Y\| = 1 \text{ a.s.} \quad \text{and} \quad E[\|G(Y)\|^2] < c.
\]

It is therefore possible to construct a sequence \( \{Y_k\}_{k \in \mathbb{N}} \) of independent symmetric pregaussian \( B \)-valued random variables with corresponding sequence of independent Gaussian random variables \( \{G(Y_k)\}_{k \in \mathbb{N}} \) such that:

\[
\|Y_k\| = 1 \text{ a.s. for every } k \quad \text{and} \quad \sum_{k \in \mathbb{N}} E[\|G(Y_k)\|] < \infty.
\]

Consider now the \( l_p(B) \)-valued \( (1 \leq p < 2) \) random variable \( Z = \{Z^k\}_{k \in \mathbb{N}} \) defined by: for every \( k \), \( Z^k = \frac{1}{k^{1/2p}} Y^k \mathbbm{1}_{\{N^2 \leq k < N^2 + N\}} \) with \( N \) being an integer valued random variable independent of what precedes defined by:

\[
P\{N = k\} = \frac{d}{k^{1+\alpha}} \quad (d = d(\alpha) > 0)
\]

where \( 0 < \frac{\alpha}{1+\alpha} < 1 - \frac{p}{2} \). It is easily seen that \( Z \) is pregaussian and bounded.

Moreover, arguing as in [2], one obtains for every \( k \) and every \( n \) sufficiently large:

\[
E[\max_{1 \leq j \leq n} \|Z_j^k\|^p] = \frac{1}{\sqrt{k}} E[\max_{1 \leq j \leq n} I_{\{N_j^2 \leq k < N_j^2 + N\}}] - (1 - P\{N^2 \leq k < N^2 + N\})^n
\]

where \( \{N_j\}_{j \in \mathbb{N}} \) is a sequence of independent copies of \( N \); further:

\[
E[\max_{1 \leq j \leq n} \|Z_j^k\|^p] = \frac{1}{\sqrt{k}} (1 - (1 - P\{N^2 \leq k < N^2 + N\})^n)
\]
\[
\sum_{k \in \mathbb{N}} E\left[ \max_{1 \leq j \leq n} \frac{\|Z_k\|}{\sqrt{n}} \right] \geq \frac{1}{2} n^{-p/2} \sum_{j \in \mathbb{N}} \left(1 - P\{ N \neq j \}^n \right)
\]

where \( I(n) = \{ j \in \mathbb{N} : 2n P\{ N = j \} \leq 1 \} \). The proof of the lemma and the definition of \( N \) thus indicate that:

\[
\sup_{n \in \mathbb{N}} E\left[ \frac{\|S_n(Z)\|}{\sqrt{n \log n}} \right] = \infty .
\]

Consider now a strictly increasing sequence \( \{p_k\} \in \mathbb{N} \) of real numbers converging to 2. Since \( p_k < 2 \) for each \( k \), by the above construction one can find a sequence \( (Z(k))_{k \in \mathbb{N}} \) of independent (pregaussian) random variables such that, for every \( k \), \( Z(k) \) takes its values in \( \mathcal{A}_{p_k} (B) \), \( \|Z(k)\| \leq 2^{-k} \) a.s., \( E[\|G(Z(k))\|] \leq 2^{-k} \) where \( G(Z(k)) \) is Gaussian with the same covariance as \( Z(k) \), and:

\[
\sup_{n \in \mathbb{N}} E\left[ \frac{\|S_n(Z(k))\|}{\sqrt{n \log n}} \right] = \infty .
\]

Now, if \( k = 1 \), there exists an integer \( n_1 \) such that:

\[
E\left[ \frac{\|S_{n_1}(Z(1))\|}{\sqrt{n_1}} \right] \geq 2 \sqrt{\log n_1}
\]

and, by dominated convergence, an integer \( m_1 \) such that if \( Z(1,m_1) \) is the projection of \( Z(1) \) onto \( \mathcal{A}_{p_1}(B) \) one has:

\[
E\left[ \frac{\|S_{n_1}(Z(1,m_1))\|}{\sqrt{n_1}} \right] \geq \sqrt{\log n_1}.
\]
Iterating the procedure, there exist strictly increasing sequences \((n_k)_k \in \mathbb{N}\) and \((m_k)_k \in \mathbb{N}\) of integers such that, if \(Z(k, m_k)\) is the restriction of \(Z(k)\) on \(l^2_p(B)\), for every \(k\):

\[
\mathbb{E}\left[ \frac{\|s_{n_k}(Z(k, m_k))\|}{\sqrt{n_k}} \right] \geq \sqrt{\log n_k}.
\]

Let now \(E = l^2_2((l^2_p(B))_k \in \mathbb{N})\) and define \(X = (X^k)_k \in \mathbb{N}\) taking values in \(E\) by \(X^k = Z(k, m_k)\) for each \(k\). \(X\) is pregaussian and bounded and \(E\) is of cotype \(2 + \delta\) and type \(2 - \delta\) for every \(\delta > 0\) but neither of cotype 2 nor type 2 since \(X\) does not satisfy the CLT for:

\[
\sup_{n \in \mathbb{N}} \mathbb{E}\left[ \left( \frac{\|s_n(X)\|}{\sqrt{n}} \right)^2 \right] = \sup_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \mathbb{E}\left[ \left( \frac{\|s_n(Z(k, m_k))\|}{\sqrt{n}} \right)^2 \right] \geq \sup_{k \in \mathbb{N}} (\log n_k) = \infty.
\]

**Remark.** To avoid the use of an abstract Banach space \(B\) of type 2 and cotype \(q\) for every \(q > 2\), minor modifications can lead to \(E = l^2_2((l^2_p(B))_k \in \mathbb{N})\) for some well-chosen strictly increasing sequences of integers \((m_k)_k \in \mathbb{N}\), \((n_j)_j \in \mathbb{N}\) and strictly increasing (resp. decreasing) sequence of real numbers \((p_k)_k \in \mathbb{N}\) (resp. \((q_j)_j \in \mathbb{N}\)) converging to 2; \(E\) is of cotype \(2 + \delta\) and type \(2 - \delta\) for every \(\delta > 0\) and there exist in \(E\) bounded pregaussian random variables which do not satisfy the CLT.

**References.**


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ON DIFFERENT VERSIONS OF THE LAW OF ITERATED LOGARITHM FOR $\mathbb{R}^\infty$ AND $l_p$ VALUED WIENER PROCESS

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Abstract. In this article we extend the laws of iterated logarithm established by M. Csorgo and P. Revesz for the standard Wiener process to $l_p$ ($p > 1$) and $\mathbb{R}^\infty$ valued Wiener processes $\{W_t; t > 0\}$. In particular one of the obtained results states that

$$\limsup_{T \to \infty} \sup_{0 < t < T - a_T} b_T (W_{t+s} - W_t) = 1 \quad \text{a.e.,}$$

where $b_T = (2a_T [\log(T/a_T) + \log \log T])^{-1/2}$, while $a_T$ is an nondecreasing function of $T$.

1. Introduction

Let $E$ denote a real separable Frechet space, $E^*$ its dual space and $(\Omega, \Sigma, P)$ a probability space. Moreover, let $\mathbb{R}^\infty$ denote the space of all real sequences $x = (x_k)_{k=1}^\infty$ with a seminorm defined by

$$(1) \quad \|x\| = \sum_{k=1}^\infty |x_k|(1 + |x_k|)^{-1},$$

and let $l_p (p > 1)$ denote the space of all real sequences $x = (x_k)_{k=1}^\infty$ such that $\sum_{k=1}^\infty |x_k|^p < \infty$ with norm

$$(2) \quad \|x\|_p = (\sum_{k=1}^\infty |x_k|^p)^{1/p}.$$

For the convenience of the readers we recall the definition and a representation of $E$ valued Wiener process which are basic in our consideration. (for more information see [12] and [13]).

Definition 1.

A family of functions $\{W_t; t > 0\}$, where $W_t : \Omega \to E$ for each $t > 0$, is called an $E$ valued Wiener process if:
(i) for any \( \{y_1, \ldots, y_n\} \subseteq E^* \) and \( \{t_1, \ldots, t_n\} \subseteq [0, \infty) \)

the random variable \( \sum_{i=1}^{n} y_i(W_{t_i}) \) has a gaussian distribution and

(ii) for all \( t, s > 0 \), and \( x, y \in E^* \) there exists a positive defined operator \( R \) such that

\[
\text{Ex}(W_t) y(W_s) = \langle y, R x \rangle \min(t, s).
\]

Proposition 1 given below shows a method of construction an \( E \) valued Wiener process. The standard proof is based on the well known generalized Ito - Nisio theorem. More detailed information are contained in [12] and [13].

Proposition 1.

Let \( g_n \) be a fundamental sequence in \( L^2 \) and let \( \{W_t; t \geq 0\} \) be an \( E \) valued Wiener process. Then there exists a sequence of \( E \) valued independent, symmetric, gaussian random variables \( Z_n \) with a common law such that

\[
W_t = \sum_{n=1}^{\infty} Z_n \int_{0}^{t} g_n(s) ds \quad \text{for} \quad t > 0.
\]

The series (3) is a.e. convergent in the norm of \( E \). If the considered Wiener process \( \{W_t; t > 0\} \) is \( l_p \) valued, then \( \| \cdot \| \) stands for the norm defined by (2) and if \( \{W_t; t > 0\} \) is a \( R^m \) valued processes, then \( \| \cdot \| \) is the norm defined by (1).

2. Chung's version of the law of iterated logarithm

The following analogue of Chung's law of iterated logarithm established by N. Jain and W. Pruitt in [8] permits to approximate the maximum of the \( l_p \) and \( R^m \) valued Wiener process by the constant \( \pi/\sqrt{8} \).

Let \( \{W_t, t > 0\} \) be a Wiener process (either \( l_p, p > 1 \) or \( R^m \) valued). For \( t > 0 \) let

\[
R_t = \max_{0 \leq s \leq t} \| W_s \|, \quad \text{where} \quad 0 < s < t
\]

and for \( n = 1, 2, \ldots \) let \( t_n = cn^n \), where \( c > 0 \).

Theorem 1.

Under the adopted assumptions

\[
\lim_{n} \inf \left( t_n / \log \log t_n \right)^{-1/2} R_{t_n} < \pi/\sqrt{8} \quad \text{a.e.}
\]

Proof. First we consider the \( l_p \) case and we show that
(5) \[ P[R_t < x] < (4/\pi)\exp(-\pi^2/8x^2) \]

and that

(6) \[ P[R_t < x] > (4/\pi)[\exp(-\pi^2 N^2/8x^2) - 1/3 A], \]

where \( A = \exp(-9N^2\pi^2/8x^2) \),

where \( N \) is a positive integer constant.

If \( W_t = (W_t^1, \ldots) \), where \( W_t^i \) is the \( n \)-th coordinate of \( W_t \), then by [6] for any \( i = 1, 2, \ldots \) we obtain the following result:

if \( R_t^i = \max_{0 < s < t} |W_s^i| \), then

\[
P[R_t^i < x\sqrt{t}] = (4/\pi) \sum_{n=1}^{\infty} (-1)^n (2n + 1)^{n-1} \exp(-\pi^2(2n + 1)^2/8x^2).
\]

Thus

\[
P[R_t^i < x\sqrt{t}] < (4/\pi)\exp(-\pi^2/8x^2)
\]

and

\[
P[R_t^i < x\sqrt{t}] > (4/\pi)[\exp(-\pi^2/8x^2) - 1/3 A].
\]

To show (5) it is sufficient to note that if \( \|W_t\|_p < x \), then \( |W_t^i| < x \) for any positive integer \( i \).

This leads to

\[
P(\max_{0 < s < 1} |W_s^i| < x) < P(\max_{0 < s < 1} |W_s^i| < x) < (4/\pi)\exp(-\pi^2/8x^2)
\]

and completes the proof of (5).

We have

\[
P(\max_{0 < s < 1} \sum_{n=1}^{\infty} |W_s^n|_p < x^p) = 1 - P(\max_{0 < s < 1} \sum_{n=1}^{\infty} |W_s^n|_p > x^p).
\]

Because \( \|W_s\|_p > x^p \), we can find a positive integer constant \( N \) such that \( |W_s^i| > x/N \) for \( i = 1, \ldots, N \).

Hence

\[
P(\max_{0 < s < 1} \sum_{n=1}^{\infty} |W_s^n|_p < x^p) > 1 - P(\max_{0 < s < 1} |W_s^i| > x/N) =
\]

\[
P(\max_{0 < s < 1} |W_s^i| < x/N) > (4/\pi)[\exp(-\pi^2 N^2/8x^2) - 1/3A]
\]

and (6) follows.
Now we prove Theorem 1.

Let $U_n = \max_{t_{n-1} < s < t_n} |W_s - W_{t_{n-1}}|_p$. Then (cf. [8])

$$P\{U_n < \left(\frac{\pi}{\sqrt{8}}\right)\left(\frac{t_n}{\log \log t_n}\right)^{1/2}\} > (\pi \cdot n \log n)^{-1}.$$

Using the Borel–Cantelli lemma leads to

$$\liminf_n \left(\frac{t_n}{\log \log t_n}\right)^{-1/2} U_n < \frac{\pi}{\sqrt{8}}.$$

Since $R_n < R_{n-1} + U_n$, we obtain

$$\limsup_n \left(\frac{t_n}{\log \log t_n}\right)^{-1/2} R_n = \limsup_n \left(\frac{t_n}{\log \log t_n}\right)^{1/2} \frac{\left(\frac{t_{n-1}}{\log \log t_{n-1}}\right)^{-1/2}}{\left(\frac{t_n}{\log \log t_n}\right)^{1/2}} R_{n-1}.$$

Taking into account the form of $t_n$ and the classical law of the iterated logarithm we get (4).

To show that (5) and (6) remain valid in the $R^\infty$ case note that if $|W_i|_\infty > a$, then there exists a positive integer constant $N$ such that $|W_i|_\infty > a/N$ for $i = 1, \ldots, N$. We obtain the assertion substitution the $1$–$1_p$ norm by $1$–$\infty$ norm in (5) and (6).

3. Csorgo–Revesz version of the LIL

In this section it is shown that the results established by M. Csorgo and P. Revesz in [4] remain valid for $1_p$ and for $R^\infty$ valued Wiener processes. In particular we show that Theorem 3 in [4] is still valid in the considered case.

First we introduce some notations.

Let $a_T$ be a nondecreasing function of $T > 1$ such that

(i) $0 < a_T < T$

(ii) $a_T/T$ is nonincreasing.

Let

(iii) $b_T = (2a_T[\log(T/a_T) + \log \log T])^{-1/2}$.

Theorem 2. Let $\{W_t : t > 0\}$ be either $1_p$ or $R^\infty$ valued Wiener process. Then

$$\limsup_{T \to \infty} \sup_{0 < t \leq T - a_T} \sup_{0 < s < a_t} b_T ||W_{t+1} - W_t|| = 1$$ a.e.
To prove this theorem we need the following two lemmas.

Lemma 1.

For any $\varepsilon > 0$ there exists a positive constant $C > 0$ such that

$$(7) \quad P\{ \sup_{0<s,s+t<h} \sup_{0\leq t<h} \| W_{s+t} - W_s \| > \sqrt{v h} \} < C h^{-1} \exp(-v^2/(2 + \varepsilon)),$$

where $v$ and $h$ are some arbitrary numbers, while $N$ is a positive integer constant.

Proof.

Let $K = W_{s+t} - W_s$ and let $K^i = W^i_{s+t} - W^i_s$, where $W^i$ is defined as in Theorem 1. Using the proof of the inequality (6) we have the following result:

if $\|K\|_p > \sqrt{v h}$, then there exists a positive integer constant $N$ such that $|K^i| > \sqrt{v h} N^{-1}$ for $i = 1, 2, \ldots, N$. Using this we obtain

$$P\{ \sup_{0<s,s+t<h} \sup_{0\leq t<h} \| K^i \| > \sqrt{v h} \} < C h^{-1} \exp(-v^2/(2 + \varepsilon)).$$

Lemma 2.

For each $\varepsilon > 0$ there exists a positive constant $C > 0$ such that

$$(7) \quad P\{ \sup_{0<s,s+t<T} \sup_{0\leq t<T} \| W_{t+s} - W_t \| > \sqrt{v h} \} < C T^{-1} \exp(-v^2/(2 + \varepsilon)),$$

where $v, h$ and $T$ are arbitrary numbers, but $N$ is a positive integer.

The desired inequality follows from the Lemma 1 and from the homogeneity of the coordinates of $\{W_t; t > 0\}$.

Proof of Theorem 2.

Step (i). Let

$$L_T = \sup_{0\leq t\leq T} \sup_{0<s,T-s \leq T} |W_{t+s} - W_t|_p.$$

We show that

$$(8) \quad \lim_{T \to \infty} \sup_{T} L_T < 1 \quad \text{a.e.}$$

From Lemma 2 it follows that

$$P\{L_T > 1 + \varepsilon\} < C (a_T/T)^{\varepsilon} (\log T)^{-(1+\varepsilon)}.$$ 

For $Q > 1$ let $T_k = Q^k$; $k = 1, 2, \ldots$. We have that

$$\sum_{k=1}^{\infty} P\{L_{T_k} > 1 + \varepsilon\} < \infty \quad \text{for any } \varepsilon > 0.$$
so that

$$\limsup_{k \to \infty} l_{n_k} \leq 1 \text{ a.e.}$$

To complete the proof of Step (i) note that for \( k \) sufficiently large

$$1 < b_{T_k} / b_{T_{k+1}} < Q$$

and \( b_{T_k} l_{T_k} \) is nonincreasing.

Step (ii).

Put \( B_T = b_T \| W_T - W(T - a_T) \| p \).

Then

$$\limsup_{T \to \infty} B_T > 1 \text{ a.e.}$$

To show this inequality note that for large \( T \) (cf. [4])

$$P(B_T > 1 - \varepsilon) > \frac{\exp(-(1 - \varepsilon)^2[\log(T/a_T) + \log \log T])}{\sqrt{2\pi}[\log(T/a_T) + \log \log T]^{1/2}} \cdot \left(\frac{a_T}{T \log T}\right)^{1-\varepsilon}.$$ 

Now let \( r = \lim a_T/T \) and let \( T_1 = 1 \).

If \( r < 1 \), then put \( T_{k+1} - Q_{T_k} = T_k \), elsewhere \( T_{k+1} = Q+1 \), where \( q \) is a positive constant.

In case \( r < 1 \) the random variables \( B_{T_n} \) are independent and we have the needed inequality.

In case \( r = 1 \) we have \( a_T > T_{k+1} - T_k \) and we obtain

$$B_{T_{k+1}} = b_{T_{k+1}} \| W_{T_{k+1}} - W(T_{k+1} - a_{T_{k+1}}) \| p.$$ 

The last inequality follows from
From Step (i) we have

\[ P\{ \sup_{k+1 \leq \ell \leq k+\infty} \| \mathcal{W}_\ell - \mathcal{W}_k \| \geq 1 - \varepsilon \} = O(k^{-(1-\varepsilon)^2 Q/Q-1}) \]

Easily we have

\[ \limsup \sup_{k+1 \leq \ell \leq k+\infty} \| \mathcal{W}_\ell - \mathcal{W}_k \| \leq 1. \]

From (9) and (10) we obtain the needed result. This completes the proof of the Step (ii). Lemmas 1 and 2 and the other results established here for \( L_p \) valued Wiener process remain valid in \( \mathbb{R}^m \) case. This completes the proof of the Theorem 2.

In Theorem 2 we estimate the upper limit of the normalized increments of a \( L_p \) and \( \mathbb{R}^m \) valued Wiener process. Under an additional assumption concerning the rate of growth of \( \log \log T \) we may replace in Theorem 2 the upper limit by the usual one.

Assume that

\[ \log \frac{T}{a_T} \rightarrow \infty. \]

(iv)

\[ \lim_{T \to \infty} \frac{\log \log T}{\log \log \log T} = 0. \]

We have the following

Theorem 3. Under the conditions (i) - (iv)

\[ \limsup_{T \to \infty} \sup_{s \leq a_T} \| \mathcal{W}_{T-s} - \mathcal{W}_s \| = 1 \quad \text{a.e.} \]

Proof.

Let \( C_T = \sup_{s \leq a_T} \| \mathcal{W}_{T-s} - \mathcal{W}_s \| \).

It is sufficient to show that \( \liminf_{T \to \infty} C_T > 1 \).

Note that

\[ X_k = b_T \| \mathcal{W}_{(k+1)a_T} - \mathcal{W}_{ka_T} \| , \quad k = 1, 2, \ldots \]

are independent. Applying the inequality from Step (ii) we have

\[ P\{ \max_{0 \leq k \leq [T/a_T]-1} b_T \| \mathcal{W}_{(k+1)a_T} - \mathcal{W}_{ka_T} \| < 1 - \varepsilon \} = \]
Using (iv) we obtain
\[ \sum_{j=1}^{\infty} \exp(-j(a_j)(\log j)^{-(1+\epsilon)}) < \infty. \]

Obviously, we have

\[ \liminf_{j \to \infty} C_j > \liminf_{j \to \infty} \max_{0 \leq k \leq j-1} b_{j}W((k+1)a_{j}) - W_{k}a_{j} \geq 1. \]

Then
\[ C_T > \max_{0 \leq k \leq j} b_{j+1}W((k+1)a_{j}) - W_{k}a_{j} \]
\[ \sup_{0 \leq t < T} \sup_{0 \leq s \leq t} b_T W_{t+s} - W_{s}. \]

Applying Step (i), Step (ii) and (11) the desired result is obtained. Similarly, as before, we can repeat the same consideration for $\mathbb{R}^\infty$ case. This ends the proof of the theorem.

4. Conclusions

Due to the general form of the normalizing constant $b_T$ several conclusions follow from the proved theorems. In particular: the classical law of the iterated logarithm, Erdos-Renyi theorem and the strong law of large numbers (under the additional assumption concerning the exponential moment) follow easily from Theorem 3. From the other side this Theorem leads to the law of iterated logarithm for any separable Hilbert space valued Wiener process.

Remarks.

1. Let $X, X_1, \ldots$ be a sequence of $L_p$ or $\mathbb{R}^\infty$ valued independent, identically distributed random vectors. Let $S_n = X_1 + \ldots + X_n$. Assume that

(i) $E X^2 = 0$, where $X^i$ is $i$-th coordinate of $X$,

(ii) there exists a number $t_0 > 0$ such that
\[ E \exp t|X| < \infty \quad \text{for} \; t < t_0. \]

Then
\[ \limsup_{n \to \infty} \sup_{k \in \mathbb{N}} n^{\frac{1}{2n}} \left( S_{k+n}^a - S_k^a \right) = 1 \quad \text{a.e.,} \]

where \( b_n = \left( 2n \log \log n \right)^{-1/2} \)

2. From the Parseval equality follows that any separable Hilbert space may be identified with \( l_2 \). All results obtained in Section 2 and 3 remain valid in this case. In particular Theorem 3 states that the law of iterated logarithm holds for any Wiener process with values in a separable Hilbert space.

3. After this work was completed, the author was made aware of the work of A. de Acosta and J. Kuelbs "Limit theorems for moving averages of independent random vectors" Z. Wahrsch. verw. Gebiete 64(1983), 67-123, from which one can deduce the law of iterated logarithm for any Wiener process with values in a separable Banach space. However our method presented here permits to obtain such law for some non-Banach sequence spaces as \( R^p \) or \( l_p \), where \( 0 < p_n < 1 \), cf. [16].

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EXTENSIONS OF THE SLEPIAN LEMMA TO p-STABLE MEASURES

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The aim of this paper is to show that the Slepian lemma fails in the case of p-stable symmetric measures on \( \mathbb{R}^n \) provided that \( 1 < p < 2 \) and \( n > 2 \) or \( 0 < p \leq 1 \) and \( n > 2(2/p) - 1 \).

1. Introduction

Let \( \mu \) be a probability measure (p.m.) on \( \mathbb{R}^n \). Its characteristic function \( \hat{\mu} : \mathbb{R}^n \rightarrow \mathbb{C} \) (field of complex numbers) is defined by

\[
\hat{\mu}(a) := \int_{\mathbb{R}^n} \exp(i \langle \alpha, \xi \rangle) d\mu(x), \ a \in \mathbb{R}^n.
\]

Given two p.m.'s \( \mu \) and \( \nu \) on \( \mathbb{R}^n \) we write \( \mu \leq \nu \) provided that

\[
|1 - \hat{\mu}(a)| \leq |1 - \hat{\nu}(a)|
\]

for each \( a \in \mathbb{R}^n \). We shall say that \( \mu \) is weakly dominated by \( \nu \) in this case.

A symmetric p.m. \( \mu \) on \( \mathbb{R}^n \) (\( \mu(B) = \mu(-B) \) for each Borel set \( B \)) is said to be p-stable, \( 0 < p \leq 2 \), if

\[
\hat{\mu}(a) = \exp(-\gamma(a)), \ a \in \mathbb{R}^n,
\]

where \( \gamma : \mathbb{R}^n \rightarrow [0, \infty) \) satisfies \( \gamma(ta) = |t|^p \gamma(a), \ t \in \mathbb{R}, a \in \mathbb{R}^n \). The set of 2-stable symmetric measures coincides with the set of Gaussian symmetric measures. For more informations about p-stable measures we refer to [7].
The following lemma is easy to establish.

**Lemma 1** Let $\mathcal{M}$ and $\mathcal{N}$ be two $p$-stable symmetric measures on $\mathbb{R}^n$. Then the following are equivalent:

(i) $\mathcal{M} \leq \mathcal{N}$

(ii) For some (each) real number $r \in (0, p)$ we have

$$\int_{\mathbb{R}^n} |x, a|^r d\mathcal{M}(x) \leq \int_{\mathbb{R}^n} |x, a|^r d\mathcal{N}(x), \quad a \in \mathbb{R}^n.$$ 

(iii) If $a \in \mathbb{R}^n$, then

$$\mathcal{M}\{x \in \mathbb{R}^n; |x, a| \leq 1\} \geq \mathcal{N}\{x \in \mathbb{R}^n; |x, a| \leq 1\}.$$ 

**Remark.** Note that (iii) has a geometrical meaning. It tells us that the $\mathcal{N}$-measure of every symmetric strip is less than its $\mathcal{M}$-measure.

The starting point of our investigations was the following result due to D. Slepian (cf. [13] or [3]).

**Lemma 2 (Slepian lemma)** Let $\mathcal{M}$ and $\mathcal{N}$ be two 2-stable (Gaussian) symmetric measures on $\mathbb{R}^n$ such that $\mathcal{M} \leq \mathcal{N}$. Then $\mathcal{M}(A) \geq \mathcal{N}(A)$ for each absolutely convex closed subset $A \subseteq \mathbb{R}^n$.

**Remark.** In other words, if $\mathcal{M}$ and $\mathcal{N}$ are Gaussian and satisfy (iii) of Lemma 1, then the estimation remains true in the case of intersections of symmetric strips.

2. **The Slepian lemma in the case of $p$-stable measures, $p < 2$**

The aim of this section is to show that the Slepian lemma fails for $p$-stable measures with $p < 2$. Before we need two easy and well-known lemmas. We include the proofs for the sake of completeness.

**Lemma 3** If $1 \leq p < 2$ and $0 \leq t \leq 1$, then

$$1 + t^p \leq 2^{1-p} \left[ (1+t)^p + (1-t)^p \right].$$
Proof. Since the case $p=1$ is obviously true we may assume that $1 < p \leq 2$. Setting

$$F(t) := (1+t)^p + (1-t)^p - 2^{p-1}(1+t^p)$$

and

$$G(t) := \frac{F(t)}{t^p}$$

it follows that

$$G'(t) = pt^{p-1} \left[ (1+t)^{p-1} + (1-t)^{p-1} - 2^{p-1} \right].$$

Since $0 < p-1 < 1$ we have

$$2^{p-1} = \left[ (1+t)^{p-1} + (1-t)^{p-1} \right] = 1 + t^p \quad 0 < t < 1,$$

i.e. $G'(t) \leq 0$, $0 < t < 1$.

Because of $G(1) = 0$ we conclude $G(t) \geq 0$ and by definition we have $F(t) \geq 0$, $0 \leq t \leq 1$, as asserted.

**Lemma 4.** For any pair $\alpha, \beta$ of real numbers and each $p \in [1,2]$ we have

$$|\alpha|^p + |\beta|^p \leq \frac{2^{1-p}}{2^{p-1}} \left[ |\alpha + \beta|^p + |\alpha - \beta|^p \right].$$

Proof. Since $2^{1-p} \geq 1$ the estimation is true in the case $\alpha = 0$. Consequently, the asserted inequality is equivalent to

$$1 + t^p \leq \frac{2^{1-p}}{2^{p-1}} \left[ |1+t|^p + |1-t|^p \right], \quad t \in \mathbb{R}.$$

But this is an easy consequence of the inequality in Lemma 3. Indeed, if $0 \leq t \leq 1$, then we may use Lemma 3. The remaining cases follow from this by replacing $t$ by $-t$ ($t < 0$) or by replacing $t$ by $t^{-1}$ ($1 < t < \infty$) and multiplying both sides with $t^p$.

**Remark.** Lemma 4 can also be proved by dualizing the first Clarkson inequality for $p \in [2,\infty)$ (cf. [14]).

Now we are in position to construct the announced counter example in the case $1 < p < 2$.

**Proposition 5.** Suppose $1 < p < 2$. Then there are two $p$-stable symmetric measures $\mu$ and $\nu$ on $\mathbb{R}^2$ with $\mu \ll \nu$ and absolutely convex closed subsets $A \subseteq \mathbb{R}^2$ such that $\mu(A) < \nu(A)$.
Proof. Define elements $x_1, x_2$ and $y_1, y_2$ of $\mathbb{R}^2$ by

$$x_1 := (2^{1-(1/p)}, 0), \quad x_2 := (0, 2^{1-(1/p)}), \quad y_1 := (1, 1) \quad \text{and} \quad y_2 := (1, -1).$$

Let $\mu$ and $\nu$ be the $p$-stable symmetric measures on $\mathbb{R}^2$ possessing the characteristic functions

$$\widehat{\mu}(a) = \exp(-Kx_1, a)^p - Kx_2, a)^p) \quad \text{and} \quad \widehat{\nu}(a) = \exp(-Ky_1, a)^p - Ky_2, a)^p)$$

for all $a \in \mathbb{R}^2$, i.e. $\mu \geq \nu$.

To complete the proof we assume that $\mu(A) > \nu(A)$ for all absolutely convex closed subsets of $\mathbb{R}^2$. Consequently, if $\| \cdot \|$ is any norm on $\mathbb{R}^2$, we have

$$\mu\{x \in \mathbb{R}^2; \|x\| > t\} \leq \nu\{x \in \mathbb{R}^2; \|x\| > t\}$$

for any $t > 0$. Particularly, this implies

$$\lim_{t \to \infty} t^p \mu\{\|x\| > t\} \leq \lim_{t \to \infty} t^p \nu\{\|x\| > t\}. $$

Note that by [1] this limit always exists. Moreover, it is known (cf. [2]) that

$$\lim_{t \to \infty} t^p \mu\{\|x\| > t\} = c_p (\|x_1\|^p + \|x_2\|^p)$$

and

$$\lim_{t \to \infty} t^p \nu\{\|x\| > t\} = c_p (\|y_1\|^p + \|y_2\|^p)$$

for some universal constant $c_p > 0$. Consequently, it follows that

$$\|x_1\|^p + \|x_2\|^p \leq \|y_1\|^p + \|y_2\|^p$$

for any norm $\| \cdot \|$ on $\mathbb{R}^2$. But, if $1 < p < 2$, then this is surely false. For instance one may take the norm

$$\|x\|_\infty := \max(\|x_1\|, \|x_2\|), \quad x = (x_1, x_2) \quad \text{or} \quad \|x\|_q := (\|x_1\|^q + \|x_2\|^q)^{1/q}$$

with $p' < q < \infty$, $1/p + 1/p' = 1$.

This contradiction ends the proof.

Next we treat the case $0 < p < 1$. We start with a result valid for all $p \in (0, 2)$. 

PROPOSITION 6. Fix $p \in (0,2)$ and let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ possessing the following property: if $\mu$ and $\nu$ are $p$-stable symmetric measures on $\mathbb{R}^n$ with $\mu \leq \nu$, then

$$\mu\{\|x\| > t\} \leq \nu\{\|x\| > t\}, \quad t > 0.$$ 

In this case $\mathbb{R}^n$ provided with this norm embeds isometrically into a suitable space $L_p$.

**Proof.** Let $x_1, \ldots, x_m, y_1, \ldots, y_m$ be arbitrary elements of $\mathbb{R}^n$ such that

$$\sum_{j=1}^m k x_j, a \chi^p \leq \sum_{j=1}^m k y_j, a \chi^p$$

for each $a \in \mathbb{R}^n$. Then we define $p$-stable symmetric measures $\mu$ and $\nu$ on $\mathbb{R}^n$ by

$$\mu(a) = \exp\left(- \sum_{j=1}^m k x_j, a \chi^p\right), \quad a \in \mathbb{R}^n,$$

$$\nu(a) = \exp\left(- \sum_{j=1}^m k y_j, a \chi^p\right), \quad a \in \mathbb{R}^n,$$

respectively. Because of $(+)$ we obtain $\mu \leq \nu$ and by assumption it follows that

$$\mu\{\|x\| > t\} \leq \nu\{\|x\| > t\}, \quad t > 0.$$ 

Now we continue as in the proof of Prop. 5 and obtain

$$\sum_{j=1}^m \|x_j\|^p \leq \sum_{j=1}^m \|y_j\|^p.$$ 

Using a result of Lindenstrauss-Pełczynski ([9]) and Maurey ([11]) it follows that $[\mathbb{R}^n, \| \cdot \|]$ embeds isometrically into some $L_p$.

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables such that

$$P\{\xi_j = 1\} = P\{\xi_j = -1\} = 1/2, \quad j = 1, 2, \ldots$$

**Lemma 7.** Suppose that $[\mathbb{R}^n, \| \cdot \|]$ embeds isometrically into some $L_p$, $0 < p < 1$. Then we have

$$\left\| \sum_{j=1}^m \|x_j\|^2 \right\|^{1/2} \leq 2^{(1/p)-(1/2)} \left\{ E \left\| \sum_{j=1}^m \xi_j x_j \right\|^p \right\}^{1/p}$$

for all $x_1, \ldots, x_m \in \mathbb{R}^n$. Here $E$ denotes the mathematical expectation.
Proof. From Khinchin's inequality

\[ \left\{ \sum_{j=1}^{m} |\alpha_j|^2 \right\}^{1/2} \leq 2^{(1/p)-(1/2)} \left\{ \mathbb{E} \left[ \sum_{j=1}^{m} \xi_j \omega_j \right]^p \right\}^{1/p} , \]

\( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \), (cf. [5]) the asserted inequality follows by using estimation 22.3.1 of [12].

**Proposition 8.** If 0 < p ≤ 1, then Slepian's lemma fails for p-stable symmetric measures on \( \mathbb{R}^n \) provided that

\[ n > 2^{(2/p)-1} . \]

Proof. By virtue of Prop. 6 it suffices to show that, if \( n > 2^{(2/p)-1} \), the space \( [\mathbb{R}^n, \| \cdot \|_p] \) does not embed isometrically into \( L_p \). But this is a consequence of Lemma 7. Indeed take the unit vectors of \( \mathbb{R}^n \) to see that the estimation in Lemma 7 is not true in this case.

**Remarks.**

1° It is open whether or not Slepian's lemma is false in the case \( 2 < n < 2^{(2/p)-1} , 0 < p \leq 1 \).

2° Note that we do not know the concrete structure of p-stable measures, 0 < p ≤ 1, failing the Slepian lemma. More precisely, we do not know how many points \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \) in \( \mathbb{R}^n \) are necessary to construct such p-stable symmetric measures.

**References:**


SOME REMARKS ON ELLIPTICALLY CONTOURED MEASURES

by

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In the recent years there are more and more works about probability measures being in a way the generalizations of the measures invariant on the rotations, for example about spherically generated measures ([2]), elliptically contoured measures ([1], [4], [5]) or integral means from Gaussian measures. In this note we study relations between these classes of measures.

The main result of this note is contained in Proposition 1. It is shown that in infinite dimensional Banach spaces the set of all spherically generated measures is the weak closure of the set of all elliptically contoured measures.

In the second part we study the support and the space of admissible translates for the elliptically contoured measures. Propositions 2 and 3 show that in fact our problem can be reduced to the investigation of supports and spaces of admissible translates of corresponding Gaussian measures.

§1. Here we consider probability measures on the Banach space $E$. We say that a probability measure $\mu$ on $R^n$ is spherically generated if it is the distribution of a random vector $Y = AX$ where $A$ is a nondegenerate $n \times n$ matrix and $X$ is a random vector invariant on the rotations. If the spherically generated measure has a density then we call it elliptically contoured.

We say that a measure $\mu$ on a Banach space $E$ is spherically generated (elliptically contoured) if every $n$-dimensional projection of $\mu$ is spherically generated (elliptically contoured) measure on $R^n$. We use the notations $EOo(E)$ for spherically generated measures and $EC(E)$ for elliptically contoured measures on the Banach space $E$.

By the central limit theorem one can obtain a class of distributions of the random vectors $Y = X \sqrt{\Theta}$ where $X$ is Gaussian random vector, $\Theta$ is positive random variable, $X$ and $\Theta$ are independent. These distributions are usually called integral means of Gaussian measures,
The following theorem shows that on infinite dimensional Banach space integral means from Gaussian measures are identical with elliptically contoured measures (see [1], [4], [5]).

**Theorem.** If \( E \) is infinite dimensional Banach space then the following conditions are equivalent:
1. \( \mu \in \mathcal{E}(E) \),
2. there exist a symmetric Gaussian measure \( \mathcal{G} \) on \( E \) and a probability measure \( \lambda \) on \( (0, \infty) \) such that
   \[
   \mu(A) = \int_0^\infty \mathcal{G}\left(\frac{A}{t}\right) \lambda(\,dt) \quad \text{for every Borel set } A \subset E,
   \]
3. \( \mu \) is an integral mean of a Gaussian measure.

From the condition 2. of Theorem follows that every \( \mu \in \mathcal{E}(E) \) is uniquely determined by the pair \( (\mathcal{G}, \lambda) \). To emphasize this correspondence we write \( \mu = E(\mathcal{G}, \lambda) \).

If a measure \( \mu \) on \( E \) is defined by the formula:
\[
\mu(A) = \int_0^\infty \mathcal{G}\left(\frac{A}{\sqrt{t}}\right) \lambda(\,dt) \quad \text{for every Borel set } A \subset E \text{ where } \mathcal{G}
\]
is a probability measure on \( E \) and \( \lambda \) is a probability measure on \( (0, \infty) \) then we will write \( \mu = (\mathcal{G}, \lambda) \).

It is clear that the following holds.

**Lemma.** Suppose that for every \( n \in \mathbb{N} \) \( \mathcal{G}_n \) and \( \lambda_n \) are probability measures on the Banach space \( E \) and on \( (0, \infty) \), respectively. If the sequences \( \{\mathcal{G}_n\} \) and \( \{\lambda_n\} \) are weakly convergent to \( \mathcal{G} \) and \( \lambda \), respectively, then the sequence \( \mu_n = (\mathcal{G}_n, \lambda_n) \) is weakly convergent to \( \mu = (\mathcal{G}, \lambda) \).

Now, we are able to state and prove our result.

**Proposition 1.** Let \( E \) be the infinite dimensional Banach space and let \( \mu_n \in \mathcal{E}(E) \) for every \( n \in \mathbb{N} \). If \( \{\mu_n\} \) weakly converges to a measure \( \mu \) such that \( \mu(\{0\}) = 0 \) then \( \mu \in \mathcal{E}(E) \).

**Proof.** Let \( 0 \neq x_0^* \in E^* \) be fixed. For every \( n \in \mathbb{N} \) we can choose (see [5]) measures \( \mathcal{G}_n \) and \( \lambda_n \) such that \( \mu_n = E(\mathcal{G}_n, \lambda_n) \) and such that for the characteristic function of \( \mu_n \) we have
\[ \hat{\mu}_n(x_0^k) = \int_0^\infty \exp \left\{ -\frac{1}{2} t \right\} \lambda_n(dt). \]

Since the sequence \( \{\hat{\mu}_n\} \) is weakly convergent we obtain
\[ \hat{\mu}_n(cx_0^k) = \int_0^\infty \exp \left\{ -\frac{c^2}{2} t \right\} \lambda_n(dt) \xrightarrow{n} \hat{\mu}(cx_0^k). \]

The function \( \psi(c^2) = \hat{\mu}(cx_0^k) \) is absolutely monotonic function as a limit of absolutely monotonic functions. So there exists a positive finite measure \( \lambda \) on \( [0, \infty) \) such that
\[ \hat{\mu}(cx_0^k) = \int_0^\infty \exp \left\{ -\frac{c^2}{2} t \right\} \lambda(dt). \]

Since \( \mu \) is nondegenerate probability measure on \( E \) without an atom at zero, then \( \lambda \) is a probability measure on \( (0, \infty) \). It is easy to see that \( \{\lambda_n\} \) is weakly convergent to \( \lambda \). Now we can find two numbers \( t_0, T \), \( T > t_0 > 0, t_0 < 1 \) such that \( \lambda([t_0, T]) = 0 \) and \( \lambda_n([t_0, T]) \geq 1 - \varepsilon \) for fixed \( \varepsilon > 0 \).

The sequence \( \{\mu_n\} \) is conditionally weakly compact and we can find a compact set \( K \subset E \) such that \( aK \subset K \) for every \( 0 < a < 1 \) and such that for every \( n \in \mathbb{N} \)
\[ \mu_n(K \setminus t_0^T) \leq 1 - \varepsilon. \]

Now we obtain
\[ 1 - \varepsilon \leq \mu_n(K \setminus t_0^T) = \int_0^T \mathcal{F}_n \left( \frac{K \setminus t_0^T}{vt} \right) \lambda_n(dt) \leq \varepsilon + \int_{t_0}^T \mathcal{F}_n \left( \frac{K \setminus t_0^T}{vt} \right) \lambda_n(dt) \]
\[ \leq \varepsilon + \mathcal{F}_n(K). \]

From the above calculations follows that the sequence \( \{\mathcal{F}_n\} \) is weakly conditionally compact so it contains subsequence \( \{\mathcal{F}_{nk}\} \) weakly convergent to a Gaussian measure \( \mathcal{F} \) (it may be degenerate).

We know that sequences \( \{\mathcal{F}_{nk}\} \) and \( \{\lambda_n\} \) are weakly convergent to \( \mathcal{F} \) and \( \lambda \), respectively, then from the Lemma follows that \( \mu_{nk} = \mathcal{E}(\mathcal{F}_{nk}) \) is weakly convergent to \( \mathcal{E}(\mathcal{F}, \lambda) \).

It is clear that \( \mu = \mathcal{E}(\mathcal{F}, \lambda) \) and since \( \mu([0]) = 0 \), so the measure \( \mathcal{F} \) can't be degenerated and \( \mu \) is elliptically contoured.
From Proposition 1 it follows immediately that in infinite dimensional Banach spaces the weak closure of the set $E_0(E)$ is the set $E_0(E) \cup \{0\}$. Moreover, we obtain that $E_0(E)$ is weakly closed and

$$E_0(E) = \left\{ \alpha 0 + \beta \alpha : 0 \leq \beta \leq 1, \mu \in E_0(E) \right\}.$$ 

§2. Let $\mu$ be a probability measure on $E$. The smallest closed subset of $E$ with the full $\mu$-measure we call the support of the measure $\mu$ and denote by $\text{supp} \mu$. It is the set of all points $x \in E$ having the property that for every open neighbourhood $U$ of $x$, $\mu(U) > 0$. The space of admissible translates for $\mu$ is the space of all points $x \in E$ such that the measure $\mu_x$, defined as follows:

$$\mu_x(A) = \mu(A - x)$$

for every Borel set $A \subset E$, is absolutely continuous with respect to $\mu$. The set $A \subset E$ is called radial if for every $t > 0$ $tA = A$.

**Proposition 2.** Let $A$ be the support of a measure $\nu$ and $A$ is a radial set. If $\mu = (\nu, \lambda)$ then $\text{supp} \mu = \text{supp} \nu$.

**Proof.** We know that $A$ is a radial set, hence

$$\mu(A) = \int_0^\infty \nu \left( \frac{A}{\sqrt{t}} \right) \lambda(dt) = \int_0^\infty \nu(A) \lambda(dt) = 1.$$ 

If there exists a closed set $B \subset A$ such that $\mu(B) = 1$ then

$$1 = \int_0^\infty \nu \left( \frac{B}{\sqrt{t}} \right) \lambda(dt) = \mu(B).$$

It means that $\nu(B/\sqrt{t}) = 1$ for $\lambda$-almost every $t > 0$. So $A \subset B/\sqrt{t}$ for $\lambda$-almost every $t > 0$, and $A \subset B$. Then $A$ is the smallest closed set for which $\mu(A) = 1$, and $A$ is the support of the measure $\mu$.

**Proposition 3.** Let $\mu = (\nu, \lambda)$ be as before and let $V_\mu$ and $V_\nu$ be the spaces of admissible translates of the measure $\mu$ and $\nu$, respectively. If $V_\nu$ is the radial set then $V_\nu \subset V_\mu$.

**Proof.** Let $x \in V_\nu$ and let $A \subset E$ be such that

$$0 = \mu(A) = \int_0^\infty \nu \left( \frac{A}{\sqrt{t}} \right) \lambda(dt).$$
It means that $\gamma \left( \frac{A - x}{t^2} \right) = 0$ for $\lambda$-almost every $t > 0$. Since $x \in V_y$ and $V_y$ is radial so $\frac{x}{t} \in V_y$ for every $t > 0$ and $\gamma \left( \frac{A - x}{t^2} \right) = 0$ for $\lambda$-almost every $t > 0$. Then we obtain

$$\mu_x(A) = \mu \left( A - x \right) = \int_0^\infty \gamma \left( \frac{A - x}{t^2} \right) \lambda(dt) = 0$$

so $x \in V_\mu$.

The space of admissible translates for the Gaussian measure is equal to Reproducing Kernel Hilbert Space and the support of Gaussian measure is equal to separable linear space. Both these sets are radial then from Propositions 2 and 3 it follows immediately that

If $\mu = \mathcal{E}(t, \lambda) \in \mathcal{E}(\mathbb{S})$ then we have

$$V_t \subset V_\mu \subset \text{supp } \mu = \text{supp } \mathcal{E}$$

References


GROTHENDIECK’S INEQUALITY AND MINIMAL ORTHOGONALLY SCATTERED DILATIONS

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Introduction

It is well-known that several important results concerning the geometric properties of Banach spaces can be obtained by applying Grothendieck’s inequality (cf. e.g. [10], [11], [19]). In the present paper we are concerned with applications of Grothendieck’s inequality to dilation theorems for Hilbert space valued vector measures and certain nonstationary stochastic processes first obtained in [13] and [14], respectively. Both of these topics have been then developed further cf. e.g. S.D. Chatterji [3], S. Goldstein & R. Jajte [4], A.G. Miamee & H. Salehi [12], H. Niemi [16], M.M. Rao [21], M. Rosenberg [24], and B. Truong-Van [25]. The dilation theorems of this type have important applications to linear prediction cf. [15].

In the present paper it is shown that the existence of orthogonally scattered dilations of vector measures with values in a Hilbert space and, respectively, the existence of stationary dilations of weakly harmonizable $L^2$-processes is, in fact, equivalent to Grothendieck’s inequality when restricted to the class of all nonnegative definite Hermitean matrices. This equivalence and its connection to the 2-majorizability of Hilbert space valued vector measures allows us to calculate best possible upper bounds (in fact Grothendieck’s constants of a special type) for minimal orthogonally scattered dilations. These bounds have been obtained in a different context by A. Pietsch [19].

1. Grothendieck’s inequality for nonnegative definite Hermitean matrices

We are concerned with a special case of the fundamental inequality by Grothendieck [5], obtained by considering only the class of all nonnegative definite Hermitean matrices. In fact, we present two equivalent formulations of the following result:
Let \( F \in \mathbb{R}, \mathbb{E} \) and let \( A = \{a_{j,k}\}_{j,k=1}^n \) be a nonnegative definite Hermitean \( n \times n \) matrix of elements of \( F \). There is a constant \( K > 0 \), not depending on \( A \), \( n \) and \( H \), such that for all pairs of sequences \( x_j, y_k \in H; j,k = 1, \ldots, n \), in an arbitrary Hilbert space \( H \) over \( F \):

\[
(1) \quad \sum_{j=1}^n \sum_{k=1}^n a_{j,k} (x_j, y_k)_H \leq KM \max_{1 \leq j \leq n} \|x_j\|_H \max_{1 \leq k \leq n} \|y_k\|_H,
\]

where

\[
M = \sup \left\{ \sum_{j=1}^n \sum_{k=1}^n a_{j,k} s_j^* t_k \mid (s_j, t_k)_{\mathbb{E}} \right\},
\]

\[
\|s_j\| \leq 1, \quad \|t_k\| \leq 1, \quad j, k = 1, \ldots, n \}.
\]

The smallest possible value of \( K \) satisfying (1) for all possible nonnegative definite Hermitean matrices \( A \) is denoted by \( K_{\mathbb{E}}^F \).

According to Grothendieck's fundamental inequality, the statement (GH) holds without the restriction \( A \) be a nonnegative definite Hermitean matrix (cf. e.g. [11]). The smallest possible value of the corresponding constant \( K \) is denoted by \( K_G^F \); and it is called Grothendieck's constant. Grothendieck's constants, in the case \( H \) is restricted to be an \( N \)-dimensional Hilbert space, are denoted by \( K_G^F(N) \) and \( K_{GH}^F(N) \), \( N \geq 2 \), respectively. It is obvious that these sequences are increasing with \( N \) and

\[
(2) \quad K_G^F = \lim_{N \to \infty} K_G^F(N), \quad K_{GH}^F = \lim_{N \to \infty} K_{GH}^F(N).
\]

To formulate the first characterization equivalent to (GH) we need some notation.

In what follows \( S \) stands for a \( \sigma \)-algebra of subsets in a space \( S \); and \( \|\mu\| \) is the semivariation of a countably additive (c.a.) vector measure \( \mu : S \to H \) with values in a Hilbert space \( H \). Recall that a c.a. vector measure \( \mu : S \to H \) is orthogonally scattered if

\[
(\mu(E), \mu(E'))_H = 0 \quad \text{for all disjoint } E, E' \in S.
\]

Let \( H \) be a Hilbert space over \( F \in \mathbb{R}, \mathbb{E} \). There is a constant \( K > 0 \), not depending on \( H \), \( (S, S) \) and \( \mu \), such that for any c.a. vector measure \( \mu : S \to H \) there exist a Hilbert space \( H_0 \) over \( F \) containing \( H \) as a closed linear subspace and a c.a. orthogonally scattered vector measure \( \mu_0 : S \to H_0 \) such that

\[
\|\mu_0(S)\|_{H_0}^2 \leq K \|\mu\|^2.
\]

(A) Let \( H \) be a Hilbert space over \( F \in \mathbb{R}, \mathbb{E} \). There is a constant \( K > 0 \), not depending on \( H \), \( (S, S) \) and \( \mu \), such that for any c.a. vector measure \( \mu : S \to H \) there exist a Hilbert space \( H_0 \) over \( F \) containing \( H \) as a closed linear subspace and a c.a. orthogonally scattered vector measure \( \mu_0 : S \to H_0 \) such that

\[
\|\mu_0(S)\|_{H_0}^2 \leq K \|\mu\|^2.
\]
where \( P : H \rightarrow \text{span}\{\mu(E) | E \in S\} \) is the orthogonal projection.

The smallest possible value of the constant \( K \) having the properties \((A)\) is denoted by \( K_F^A \).

Remark. The constant \( K_F^A \) and its finite dimensional equivalents give best possible upper bounds for the minimal orthogonally scattered dilation \( \mu_o \), i.e. \( \| \mu_o(S) \|_{H_o}^2 \) is smallest possible.

Remark. The existence of an orthogonally scattered dilation \( \mu_o \) (i.e. \( \mu_o \) satisfies \((3)\)) of a Hilbert space valued (Radon) measure was first obtained in [13]. For extensions of this result cf. [3], [4], [16], [25].

Let \( G \) be a locally compact Abelian group with the dual group \( \Gamma \) and neutral element \( e_G \). Recall that a function \( r : G \times G \rightarrow \mathbb{C} \) (resp. \( r_o : G \rightarrow \mathbb{C} \)) is positive definite, if

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k r(g_j, g_k) \geq 0 \quad \text{(resp.} \quad \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k r_o(g_j^{-1} g_k) \geq 0) \]

for all \( a_j \in \mathbb{C}, \ g_j \in G, \ j = 1, \ldots, n; \ n \in \mathbb{N}. \) We say that a positive definite function \( r_o : G \rightarrow \mathbb{C} \) majorizes a given positive definite function \( r : G \times G \rightarrow \mathbb{C}, \) if the function \( R(s,t) = r_o(t^{-1}s) - r(s,t), \ s,t \in G, \) is positive definite.

(B) Let \( r : G \times G \rightarrow \mathbb{C} \) be a continuous positive definite function. There exists a constant \( K > 0 \) not depending on \( G \) and \( r \) having the following property: There exists \( M > 0 \) such that

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k r(g_j, g_k) \leq M \sup_{\gamma \in \Gamma} \left| \sum_{j=1}^{n} a_j (g_j, \gamma) \right|^2 \]

for all \( a_j \in \mathbb{C}, \ g_j \in G, \ j = 1, \ldots, n; \ n \in \mathbb{N}, \) if and only if there exists a continuous positive definite function \( r_o : G \rightarrow \mathbb{C} \) with \( r_o(e_G) \leq MK \) majorizing \( r. \)

The smallest possible constant \( K \) having the properties \((B)\) is denoted by \( K_B. \)

Remark. The existence of a continuous positive definite function \( r_o : G \rightarrow \mathbb{C} \) majorizing a given continuous positive definite function
The proofs of the following Theorems I and II are presented in Section 2.

**Theorem I.** (i) Statement (GH) holds, if and only if (A) holds; and $K_{GH}^F = K_A^F$, $F \in \{\mathbb{R}, \mathbb{E}\}$.

(ii) Statement (GH) holds for $F = \mathbb{E}$, if and only if (B) holds; and $K_{GH}^E = K_B^E$.

**Remark.** It clearly follows from the proofs of Theorems I and II that also the equalities

$$K_{GH}^F(N) = K_A^F(N), \quad K_{GH}^E(N) = K_B(N)$$

hold, when the constants $K_A^F(N)$, $K_B(N)$, $N \geq 2$, are defined in the obvious way. (Note, in the latter equality $N$ is the dimension of the reproducing kernel Hilbert space associated with $r: G \times G \to \mathbb{E}$; cf. the proof of Thm II).

**Remark.** It can be deduced e.g. from [16; §4] (cf. [3], [13], [24]) that the statement (A) and, a fortiori, (GH) is equivalent to the 2-majorizability of all Hilbert space valued c.a. vector measures (or bounded Radon measures) $\mu: S \to H$, i.e. there exists a c.a. $\nu: S \to \mathbb{R}^+$ satisfying

$$\| \int_S \phi \, d\mu \|_H^2 \leq \int_S |\phi|^2 \, dv \quad \text{and} \quad \nu(S) \leq K_A^F \|\mu\|^2$$

for all bounded $\mathbb{E}$-measurable functions $\phi: S \to F$. This indicates that it is the constant $K_{GH}^F$, which is essential in the estimate

$$\mu_2(T) = \pi_2(T) = \lambda_2(T) \leq K_{GH}^F \| T \|,$$

(which cannot be improved) for the norms $\mu_2(T)$, $\pi_2(T)$, and $\lambda_2(T)$ of a bounded Radon measure $T: C(S) \to H$, on a compact Hausdorff space $S$, in the Banach spaces of all 2-majorizable, 2-absolutely summing and Pietsch 2-integral mappings, respectively (cf. [17; Satz 45], [11; pp. 70-71], [19; Chapt. 22]).
The 2-majorizability of any bounded Hilbert space valued Radon measure was first obtained by Grothendieck [5] (cf. [18], [23]).

A recent account on different estimates of $K'_G$ has been presented e.g. by Pisier [20] and Krivine [7], [8], [9] (cf. [19; p. 308]). It should be noted that

$$K'_G \leq K'_G(2) K_G$$
and $K'_G(2) = 2^{1/2}$

(cf. [20; p. 412] and [7]). However,

$$K'_G = K'_G(2) K'_G$$
and $K'_G(2) = \pi^2/8$

(cf. Theorem II).

The exact values of all the constants $K'_G$, $K'_G(N)$, $N \geq 2$, are indicated in the next theorem.

**Theorem II.** For any $N \geq 2$

$$K'_G(N) = \frac{1}{N\rho_F(N)^2}, \quad F \in \{R,E\}$$

with (in terms of the gamma function)

$$\rho_F(N) = \frac{2\Gamma(N/2)}{\pi^{1/2}(N-1)\Gamma((N-1)/2)}, \quad \rho_E(N) = \frac{\pi^{1/2}\Gamma(N)}{2\Gamma((2N+1)/2)}$$

Moreover, $K'_G = \pi/2$, $K'_G = 4/\pi$.

**Remark.** The author wants to thank Prof. A. Pietsch for kindly pointing out that the values of the constants $K'_G$, $K'_G(N)$, $N \geq 2$, have already been calculated in a different context [19; Lemma 22.1.6 with $s=2, p=1$].

Moreover, the value of $K'_A = \pi/2$ has been calculated in the context of 2-majorizable vector measures by Rogge [23]; and the estimate $K'_G \leq \pi/2$ was obtained in the present context by Rietz [22].

2. Proofs of Theorems I and II

Theorem I is easily obtained, if the equivalence of (GH) and, respectively, (E) with the 2-majorizability of c.a. Hilbert space valued vector measures is deduced (cf. the latter remark following Theorem I).
Proof of Theorem I. (i) The implication (GH) \( \Rightarrow \) 2-majorizability and \( K^F_A \leq K^F_{GH} \) can be proved by following the proof of Theorem 3.9 in [24]; and this part of the proof is therefore omitted.

To prove that 2-majorizability \( \Rightarrow \) (GH) and \( K^F_A \leq K^F_{GH} \), suppose 
\[
A = \{a_{j,k}\}_{j,k=1}^n
\]
is a nonnegative definite Hermitean matrix.

The function \( R: \{1, \ldots, n\} \times \{1, \ldots, n\} \rightarrow F \) defined by
\[
R(j,k) = \overline{a_{j,k}}, \quad j, k \in \{1, \ldots, n\},
\]
is then positive definite. Let \( H(R) \) be the associated reproducing kernel Hilbert space and \( f: \{1, \ldots, n\} \rightarrow H(R) \) a mapping satisfying
\[
(f(j), f(k))_{H(R)} = R(k, j), \quad j, k \in \{1, \ldots, n\}
\]
(cf. [2]).

The set function
\[
(6) \quad \mu(E) = \sum_{j \in E} f(j), \quad E \subseteq \{1, \ldots, n\},
\]
is then an \( H(R) \)-valued c.a. vector measure on \( S = \{1, \ldots, n\} \)
satisfying \( \| \mu \|^2 = M \) (with the notation as in (GH)).

Let \( H \) be an arbitrary Hilbert space over \( \mathbb{F} \) and let \( x_1, \ldots, x_n, y_1, \ldots, y_n \in H \). Choose an orthonormal basis \( e_1, \ldots, e_m \) in the linear subspace in \( H \) spanned by the vectors \( x_1, \ldots, x_n, y_1, \ldots, y_n \). There then exist representations
\[
x_j = \sum_{k=1}^m c_{j,k} e_k, \quad y_j = \sum_{k=1}^m d_{j,k} e_k, \quad j = 1, \ldots, n.
\]
Define the functions \( \phi_k: S \rightarrow \mathbb{F}, \quad \psi_k: S \rightarrow \mathbb{F} \) by
\[
(7) \quad \phi_k(j) = c_{j,k}, \quad \psi_k(j) = d_{j,k}, \quad j \in S, \quad k = 1, \ldots, m.
\]
Then
\[
(x_j, x_k)_H = \sum_{h=1}^m \phi_h(j) \overline{\psi_h(k)}, \quad j, k = 1, \ldots, n,
\]
and, a fortiori,
\[
\sum_{j=1}^n \sum_{k=1}^m a_{j,k} (x_j, x_k)_H = \sum_{h=1}^m \sum_{j=1}^n \sum_{k=1}^n (f(j), f(k))_{H(R)} \phi_h(j) \overline{\psi_h(k)}
\]
Furthermore, by the 2-majorizability assumption (5),

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} k(x_j, x_k)_H \leq \sum_{h=1}^{n} \left( \int \phi_h \, d\mu, \int \psi_h \, d\mu \right)_H R
\]

\[
\leq \sum_{h=1}^{m} \left( \int \phi_h \, d\mu \right)_{H(R)} \leq \left( \sum_{h=1}^{m} \left( \int \psi_h \, d\mu \right)_{H(R)} \right)^{1/2}
\]

\[
\leq \left( \sum_{h=1}^{m} \int \phi_h^2 \, dv \right)^{1/2} \leq \left( \sum_{h=1}^{m} \int \psi_h^2 \, dv \right)^{1/2}
\]

for some nonnegative \( v \) on \( S \) with \( v(S) \leq K_A \| \mu \|^2 \). Therefore,

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} k(x_j, x_k)_H \leq K_A \| \mu \|^2 \max_{1 \leq j \leq n} \left( \sum_{h=1}^{m} |\phi_h(j)|^2 \right)^{1/2} \max_{1 \leq j \leq n} \left( \sum_{h=1}^{m} |\psi_h(j)|^2 \right)^{1/2}
\]

\[
\leq K_A M \max_{1 \leq j \leq n} \| x_j \|_H \max_{1 \leq j \leq n} \| y_j \|_H ;
\]

proving (1) with \( K = K_A^F \).

(ii) It is enough to show that (B) \( \iff \) the 2-majorizability (5) and \( K_B = K_A^F \).

Suppose (5) holds for \( F = \mathbb{C} \) and \( r: G \times G + \mathbb{E} \) is a continuous positive definite function satisfying (4). By applying a standard reproducing kernel Hilbert space argument (cf. [2]), one then obtains a Hilbert space over \( \mathbb{E} \) and a continuous mapping \( x: G + H \) with

\[
(x(g), x(g'))_H = r(g, g'), \quad g, g' \in G.
\]

Moreover, by (4)

\[
\left\| \sum_{j=1}^{n} a_j x(g_j) \right\|_H^2 \leq M \sup \left\| \sum_{j=1}^{n} a_j (g_j, \gamma) \right\|^2_{\gamma \in \Gamma}
\]

for all \( a_j \in \mathbb{E} \), \( g_j \in G \), \( j = 1, \ldots, n \); \( n \in \mathbb{N} \). It then follows from Theorem 3 by Kluvanének [6] that

\[
x(g) = \int_{\Gamma} (g, \gamma) \, d\mu(\gamma), \quad g \in G,
\]
for a uniquely determined (regular) c.a. \( H \)-valued vector measure \( \mu \) on \( \Gamma \) with \( \| \mu \|^2 \leq M \).

Let \( \nu \) be a c.a. nonnegative measure on \( \Gamma \) satisfying (5) for \( \mu \). Then for all \( a_j \in \mathbb{E}, \ g_j \in \mathcal{G}, \ j = 1, \ldots, n; \ n \in \mathbb{N} \), one has

\[
\sum_{j=1}^n \sum_{k=1}^n a_j \tilde{a}_k r(g_j, \gamma) = \left\| \sum_{j=1}^n a_j \int \left( g_j, \gamma \right) \, d\mu(\gamma) \right\|^2_H \\
\leq \int \left( \sum_{j=1}^n a_j \int g_j, \gamma \right)^2 \, d\nu(\gamma) \\
= \sum_{j=1}^n \sum_{k=1}^n a_j \tilde{a}_k r_o(g_k, g_j)
\]

with the continuous positive definite function \( r_o: \mathcal{G} \to \mathbb{E} \),

\[
r_o(\gamma) = \int \left( g, \gamma \right) \, d\nu(\gamma), \quad g \in \mathcal{G},
\]

satisfying \( r_o(e_0) = \nu(\Gamma) \leq K_A^E; \) proving that (A) \( \Rightarrow \) (B) and \( K_B \leq K_A^E \).

On the other hand, suppose (B) holds. It clearly follows from the proof of the part (i) of this theorem that it is enough to prove that (B) \( \Rightarrow \) the 2-majorizability of any \( H \)-valued vector measure on any finite set.

Thus, let \( H \) be a Hilbert space over \( \mathbb{E} \) and let \( \mu \) be an \( H \)-valued vector measure on \( S = \{1, \ldots, n\} \). Consider \( S \) as the group \( G_n = \mathbb{Z}(\text{mod } n) \) and define

\[
x(\gamma) = \int \left( \gamma, g \right) \, d\mu(g), \quad \gamma \in \Gamma_n.
\]

Since (B) holds for \( r: \Gamma_n \times \Gamma_n \to \mathbb{E}, \ r(\gamma, \gamma') = \left( x(\gamma), x(\gamma') \right)_H', \ \gamma, \gamma' \in \Gamma_n \), with \( M = \| \mu \|^2 \), there exists a positive definite function \( r': \Gamma_n \to \mathbb{E}, \)

\[
r_o(\gamma) = \int \left( \gamma, g \right) \, d\nu(g), \quad \gamma \in \Gamma_n,
\]

with \( r_o(e_0) = \nu(\Gamma_n) \leq K_B \| \mu \|^2 \), majorizing \( r \). It is obvious that \( \nu \) is the desired 2-majorant of \( \mu \), i.e. \( \nu \) satisfies (5), proving that (B) \( \Rightarrow \) (A) and \( K_A^E \leq K_B \).

Our proof of Theorem II is based on Rogge's [23] approach.

For \( n \geq 2 \) let \( E_n^F = \left\{ x \in F^n | \| x \|_{F^n} = 1 \right\} \) and let \( \nu_n \) be the normalized translation invariant measure on \( E_n^F \). A calculation by using polar coordinates gives
for all \( w, z \in \mathbb{F}^n \); here \( \text{sign} 0 = 0 \), \( \text{sign} x = x/|x| \), \( x \in \mathbb{F} \), \( x \neq 0 \). Especially,

\[
(10a) \quad n \rho \mathbb{R}(n)^2 > \frac{2}{\pi}, \quad n \geq 2; \quad \lim_{n \to \infty} n \rho \mathbb{R}(n)^2 = \frac{2}{\pi}
\]

\[
(10b) \quad n \rho \mathbb{E}(n)^2 > \frac{\pi}{4}, \quad n \geq 2; \quad \lim_{n \to \infty} n \rho \mathbb{E}(n)^2 = \frac{\pi}{4}
\]

(cf. [23; p. 253], [19; §22.1]).

The formulas (8) and (9) can be used to derive the following sharpening and extension of a result by Rogge [23; Satz 1] for \( \mathbb{F} = \mathbb{R} \) to \( \mathbb{F} = \mathbb{E} \). The proof of the lemma can be obtained by a suitable modification of Rogge's proof and it is therefore omitted (cf. [24; Lemma 3.8]).

**Lemma 1.** Let \( H \) be a Hilbert space over \( \mathbb{F} \in \{ \mathbb{R}, \mathbb{E} \} \) and let \( \mu : S \to H \) be a c.a. vector measure. Then

\[
\sum_{j=1}^{n} \left\| \int \phi_j d\mu \right\|_H^2 \leq \frac{1}{n \rho \mathbb{F}(n)^2} \sup_{\omega \in S} \left[ \sum_{j=1}^{n} |\phi_j(\omega)|^2 \right]
\]

for all bounded \( S \)-measurable functions \( \phi_j : S \to \mathbb{F}, \ j = 1, \ldots, n; n \in \mathbb{N} \).

**Proof of Theorem II.** We first show that

\[
(11) \quad k_{\mathbb{E}}(N) \leq \frac{1}{N \rho \mathbb{E}(N)^2}, \quad N \geq 2.
\]

Suppose \( A = \{ a_{j,k} \}_{j,k=1}^{n} \) is a nonnegative definite Hermitian matrix and suppose \( H \) is an \( N \)-dimensional Hilbert space over \( \mathbb{F} \) and \( x_j, y_k \in H, \ j,k = 1, \ldots, n \). Since \( H \) is \( N \)-dimensional, there exists an orthonormal basis \( \phi_1, \ldots, \phi_N \) in \( H \). Define the vector measure \( \mu \) on \( S = \{ 1, \ldots, n \} \) and the functions \( \phi_k : S \to \mathbb{F} \), \( \psi_k : S \to \mathbb{E} \), \( k = 1, \ldots, N \), according to (6) and (7), respectively. Then, as in the proof of Theorem I (i),

\[
\left| \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} (x_j^* y_k)_H \right|
\]
Moreover, since \( \| \mu \|^2 \leq M \) (with the notation as in (GH)) we get by Lemma 1

\[
\left\| \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k}(x_j, x_k) \right\| \leq \frac{M}{N \rho_F(N)^2} \sup_{E \subset B_F} \left( \sum_{j=1}^{n} \| \phi_j(\omega) \|^2 \right)^{1/2} \sup_{E \subset B_F} \left( \sum_{j=1}^{n} \| \psi_j(\omega) \|^2 \right)^{1/2}
\]

\[
\leq \frac{M}{N \rho_F(N)^2} \max_{1 \leq j \leq n} \| x_j \|_H \max_{1 \leq j \leq n} \| y_j \|_H.
\]

proving (11).

To prove the inequality opposite to (11), define the \( F^N \)-valued c.a. vector measure on \( B_F^N \) by

\[
\mu_N(E) = \frac{1}{\rho_F(N)} \int_E f \, dv_N(r),
\]

for all Borel sets \( E \subset B_F^N \). It then follows, by using analogous arguments as in [23; pp. 256-257] that \( \| \mu_N \| \leq 1 \) and

\[
\sum_{k=1}^{N} \left\| \int_{B_F^N} f_k \, dv_N \right\|^2_{F^N} = \frac{1}{N \rho_F(N)^2}
\]

with

\[
f_k(r) = (e_k, r)_{F^N}, \quad r \in B_F^N,
\]

where \( e_k, \ k = 1, \ldots, N, \) is an orthonormal basis in \( F^N \).

Since the functions \( f_k, \ k = 1, \ldots, N, \) are continuous functions on the compact Hausdorff space \( B_F^N \), there exist sequences of Borel measurable simple functions

\[
\phi_k(m) = \sum_{j=1}^{M} a_{k,m,j} X_{m,j}, \quad m \in N,
\]

approximating the functions \( f_k \) uniformly on \( B_F^N \) simultaneously for all \( k = 1, \ldots, N \). Thus

\[
\sum_{k=1}^{N} \left\| \int_{B_F^N} f_k \, dv_N \right\|^2_{F^N}
\]
\[
\lim_{m \to \infty} \sum_{k=1}^{N} \left\| \int_{E} \phi_{k,m} \, d\mu_N \right\|^2_{E_N}
\]
\[
J(m) = \lim_{m \to \infty} \sum_{j=1}^{J(m)} \sum_{k=1}^{N} (\mu_N(E_{m,j}), \mu_N(E_{m,k}))_{E_N} (a_m,j,a_m,k)_{E_N}
\]
with \(a_m,j = (a_{1,m,j}, \ldots, a_{N,m,j}) \in \mathbb{F}^N\), \(j = 1, \ldots, J(m)\); \(m \in \mathbb{N}\). Since \(A_m = \{(\mu_N(E_{m,j}), \mu_N(E_{m,k}))_{E_N} \}_{j,k=1}^{J(m)}\) is a nonnegative definite Hermitian matrix with \(M = 1\) (cf. (GH)) for all \(m \in \mathbb{N}\), it follows that

\[
K_{GH}(N) \geq \frac{1}{N^2 \rho(N)^2}.
\]

Finally, the values of \(K_{GH} \in \{\mathbb{F}, \mathbb{C}\}\) can be obtained by combining (2) and (10a-b).

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References


Dependence of Gaussian measure on covariance in Hilbert space

Gyula Pap

1. Introduction. Let \((H, \langle \cdot, \cdot \rangle)\) be a real separable Hilbert space and denote by \(\mathcal{N}_T\) a centered Gaussian measure on \(H\) with covariance operator \(T : H \to H\). We shall examine the measure of balls with centre at the origin:

\[
\mathcal{N}_T \left\{ x \in H : \|x\| < \tau \right\}, \quad \tau > 0.
\]

It is known that the function \(\tau \to \mathcal{N}_T(\|x\| < \tau)\) is very nice; for example it has a bounded derivative /see Vakhania [4]/, so we have the following estimation:

\[
\left| \mathcal{N}_T(\|x\| < \tau_1) - \mathcal{N}_T(\|x\| < \tau_2) \right| \leq c(T) \left| \tau_1 - \tau_2 \right|
\]

where the constant \(c(T)\) depends only on \(T\). The above function is investigated in the case of Banach spaces, too /cf. Paulauskas [3]/.

The purpose of this paper is to show some analytical properties of the function \(T \to \mathcal{L}(T) = \mathcal{N}_T(\|x\| < \tau)\).

We can consider this function on the set of Gaussian covariance operators, which forms a convex cone in the Banach space of nuclear operators. We denote the nuclear norm by \(\|\cdot\|_4\).
2. Some analytical properties of the function $\mathcal{F}(T) = \gamma_T(\|x\| < r)$.

It is well-known that the function $\mathcal{F}$ is continuous, but we have the following statement.

**Theorem 1.** The function $\mathcal{F}(T) = \gamma_T(\|x\| < r)$ is not uniformly continuous.

**Proof.** Denote by $\{e_n\}$ an o.n.s. in $H$ and let us consider the operators

$$T_{n, \delta} = \frac{T^2 + \delta}{n} \sum_{k=1}^{n} (\cdot, e_k) e_k, \quad n \geq 1, \quad \delta > -r^2.$$ 

Then

$$\mathcal{F}(T_{n, \delta}) = P\left(\sqrt{\frac{T^2 + \delta}{n} (\xi_n^2 + \ldots + \xi_n^2)} < r\right) = P\left(\frac{\xi_n^2 + \ldots + \xi_n^2}{n} < \frac{r^2}{r^2 + \delta}\right),$$

where $\xi_1, \ldots, \xi_n$ are i.i.d. standard normal random variables. Using the law of large numbers we get

$$\lim_{n \to \infty} \mathcal{F}(T_{n, \delta}) = \begin{cases} 1 & \text{if } -r^2 < \delta < 0 \\ 0 & \text{if } \delta > 0 \end{cases}.$$ 

We have $\|T_{n, \delta_1} - T_{n, \delta_2}\|_4 = |\delta_1 - \delta_2|$, thus for arbitrary $\delta > 0$ we can find two operators $T_1$ and $T_2$ such that

$$\|T_1 - T_2\|_4 \leq \delta \quad \text{and} \quad |\mathcal{F}(T_1) - \mathcal{F}(T_2)| \geq \frac{1}{2},$$

namely

$T_1 = T_{n, \delta_1}, \quad T_2 = T_{n, \delta_2}, \quad \text{where} \quad \delta_1 = \frac{\delta}{2}, \quad \delta_2 = -\frac{\delta}{2}$

and $n$ is so large that $\mathcal{F}(T_{n, \delta_1}) \leq \frac{1}{4}$ and $\mathcal{F}(T_{n, \delta_2}) \geq \frac{3}{4}$. 

Corollary: The function \( \mathcal{F} \) has no modulus of continuity, so e.g. there is no \( \alpha > 0 \) satisfying

\[
\left| \gamma_{T_1'}(\|x\| < \tau) - \gamma_{T_2'}(\|x\| < \tau) \right| \leq c(\tau) \|T_1 - T_2\|_4^\alpha.
\]

Remark: Obviously, if we replace nuclear norm by Hilbert-Schmidt norm, or by any other norm of this type, we get the same result.

Theorem 2.: The function \( f(T) = \gamma_T(\|x\| < \tau) \) is Fréchet-differentiable. If \( T = \sum_{k=1}^\infty \lambda_k (\cdot, v_k) v_k \) where \( \{\lambda_k\} \) and \( \{v_k\} \) are the eigenvalues and the eigenvectors of \( f \), and \( \lambda_k \neq 0 \) for \( k = 1, 2, \ldots \), then the derivative of \( f \) at the point \( T \) in the direction \( S \) has the form

\[
\mathcal{D}(f(T), S) = T(\lambda_T S),
\]

where

\[
A_T = \sum_{k=1}^\infty \left( \frac{1}{2} \lambda_k \int_0^{\frac{\pi}{2}} 2 \lambda_k \sin^2 (\tau t) \right) (\cdot, v_k) v_k,
\]

\[
\mathcal{F}(\tau^2) = \gamma_T(\|x\|^2 < \tau^2).
\]

Proof: It is known that if \( \xi_T \) is a centered Gaussian random vector in \( H \) with covariance \( T \), then the characteristic function of \( \|\xi_T\|^2 \) is

\[
E e^{i\tau \|\xi_T\|^2} = \frac{1}{\sqrt{\det(I - 2it\lambda_T)}} = \frac{1}{\sqrt{\det(I - 2itT)}},
\]
where $\{\lambda_k\}$ are the eigenvalues of $T$, and $I$ is the identity operator on $H$.

By the inversion formula we get

$$\mathcal{F}(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left\{ \frac{1 - e^{-it\tau}}{it} \frac{1}{\sqrt{\text{Det}(I - 2i\tau T)}} \right\} dt.$$ 

The function $T \rightarrow \text{Det}(I + T)$ is Fréchet-differentiable on the Banach space of nuclear operators and the derivative at the point $T$ in the direction $S$ is the following /cf. Dunford, Schwartz [2]/:

$$\mathcal{D}(\text{Det}(I + T), S) = \text{Det}(I + T) \cdot \text{Tr}((I + T)^{-1} S),$$

if $-1$ is not an eigenvalue of $T$. From this in our case we have

$$\mathcal{D}(\mathcal{F}(T), S) = \text{Tr}(A_T S)$$

where

$$A_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left\{ \frac{1 - e^{-it\tau}}{it} \frac{(I - 2i\tau T)^{-1}}{\text{Det}(I - 2i\tau T)} \right\} dt.$$ 

The operator $A_T$ is linear and bounded, since

$$\|A_T\| = \sup_{\lambda \in \Lambda} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left\{ \frac{1 - e^{-it\tau}}{it} \frac{(I - 2i\tau \lambda)^{-1}}{\text{Det}(I - 2i\tau \lambda)} \right\} dt \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{\lambda \left(1 + 4\tau^2 \lambda^2\right)^{1/4}} < \infty.$$ 

So the operator $S \rightarrow \text{Tr}(A_T S)$ is also linear and bounded /cf. Birman, Solomiak [1]/.
On the other hand, $\mathcal{D}(\mathcal{L}(T), S)$ is continuous in $T$, thus the function $\mathcal{L}$ is Fréchet-differentiable.

We can compute the eigenvalues of $A_T$ in the following way. The characteristic function of $F_T(z) = \mathcal{L}_T(n \times n < z)$ is

$$
\int_{-\infty}^{\infty} e^{izt} dF_T(z) = \frac{1}{\sqrt{\text{Det}(I-2itT)}}
$$

so

$$
\int_{-\infty}^{\infty} \text{Re} \left\{ \frac{1-e^{-it\tau^2}}{\tau^2} \right\} d\tau = \int_{-\infty}^{\infty} \text{Re} \left\{ \frac{e^{it\tau} - e^{it(\tau - \eta)}}{1-2it\tau} \right\} d\tau dF_T(z).
$$

Using the formulas

$$
\int_0^\infty \frac{\cos ax}{1+x^2} \, dx = \frac{\pi}{2} e^{-|a|}, \quad \int_0^\infty \frac{x \sin bx}{x^2 + a^2} \, dx = \frac{\pi}{2} e^{-|b|}
$$

we can compute that if $\lambda_k \neq 0$, then

$$
\int_{-\infty}^{\infty} \text{Re} \left\{ \frac{e^{itz} - e^{it(\tau - \eta)}}{1-2it\tau} \right\} d\tau = \begin{cases} 
-\frac{\pi}{2\lambda_k} e^{-\frac{z^2 - \tau^2}{2\lambda_k}} & \text{if } 0 < \tau < \pi^2 \\
0 & \text{if } \tau > \pi^2
\end{cases}
$$

Of course, $F_T(z) = 0$ if $z < 0$, so the proof is complete.

**Theorem 3.** The norm of the derivative $\| D \mathcal{L}(T) \|$ is not bounded on the surface $\{T \| T \| = \pi^2\}$.

**Proof:** If $T_{\eta \lambda} = \sum_{k=1}^{N} (\cdot, e_k) e_k$, where $\{e_k\}$ is an o.n.s. in $H$, then $F_{T_{\eta \lambda}}(z) = P\left(\lambda (\sum_{k=1}^{N} e_k) < z\right)$, where...
\( \xi_1, \ldots, \xi_n \) are i.i.d. standard normal random variables, so

\[
F_{T_{n, \lambda}}(z) = \int_0^z \frac{\exp\left(-\frac{u^2}{2\lambda}\right)}{\left(2\lambda\right)^{\frac{\lambda}{2}} \Gamma\left(\frac{\lambda}{2}\right)} \, du.
\]

Since \( \|D (T)\| = \|A_T\| \), using the method of the proof of Theorem 2 we can compute \( \|D (T_{n, \lambda})\| \). If we replace \( \lambda = \frac{\tau^2}{n} \), then \( \|T_{n, \lambda}\|_4 = \tau^2 \) and we have \( \|D (T_{n, \lambda})\| \to \infty \) for \( n \to \infty \). Thus the function \( \|D (T)\| \) is not bounded on the surface \( \{\|T\|_4 = \tau^2\} \).

Remark: In the n-dimensional case \( \|D (T)\| \) is maximum at the point \( T_n = \frac{\tau^2}{n+2} \sum_{k=1}^n (e_k e_k^\top) \) where \( \{e_k\} \) is an o.n.s., and \( \|T_n\|_4 = \frac{n \tau^2}{n+2} \) if \( T_n \) is near the surface \( \{\|T\|_4 = \tau^2\} \).
References


On Subordination and Linear Transformation of Harmonizable and Periodically Correlated Processes

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The problem of finding analytic conditions for subordination of harmonizable and periodically correlated sequences is studied. Sufficient conditions for subordination of harmonizable sequences (in the spirit of Kolmogorov's work) and a simple counter-example showing that these conditions are not necessary are given. In the case of periodically correlated sequences, which is a subclass of harmonizable sequences, necessary and sufficient conditions for subordination, mutual subordination and necessary conditions for strong subordination of such processes in terms of their associated multivariate stationary sequences are derived. The problem of finding spectral conditions such that it is possible to find a mean-convergent series for a harmonizable sequence, when it is a linear transformation of another such sequence is studied. This idea is used to find an algorithm for the linear predictor and interpolator of a periodically correlated process in the time domain.

1. Introduction.

The concept of subordination of stationary processes was introduced, studied and used in prediction of univariate stationary processes by A.N. Kolmogorov [8]. Analytic conditions for subordination in terms of the spectral measures of the processes were derived in [8]. Analogous conditions for the subordination of q-variate stationary processes were derived by M. Rosenberg [18, 19], Yu. A. Rozanov [17] and P. Masani [10] and for infinite-dimensional stationary processes by V. Mandrekar and H. Salehi [9]. In [9] and [19] the notion of subordination and its analytic characterization have been used to gain some new insight into some problems in analysis and F. Graef [7] has used this idea in optimal filtering of stationary signals.

The problem of subordination of non-stationary processes have been studied by T.N. Siraya [20, 21]. In [20] he gives conditions for subordination and strong subordination of second-order processes in terms of their covariances and corresponding reproducing kernel Hilbert spaces. In [21] conditions for subordination and strong subordination of one second-order process to another such process with orthogonal increments, in terms of the structural measure of the latter has been derived. The problem of subordination of processes which are not necessarily of second-order is recently studied by A. Weron [22].

Harmonizable and periodically correlated processes are used in many areas of application [3]. In fact, it is shown in [3] that under some general conditions the output of a linear system is a harmonizable process. For related materials one can refer to the work of J.L. Abreu [1, 2] and H. Niemi [12] and the references therein.

In this paper we give analytic conditions for subordination of harmonizable and periodically correlated processes in the spirit of Kolmogorov's work [8]. Sufficient conditions for subordination of harmonizable processes and a simple counter-example showing that these conditions are not necessary along with the problem of linear transformation of harmonizable processes is discussed in section 2. Necessary and sufficient conditions for subordination, mutual subordination and necessary conditions for strong subordination, c.f. Definition 1.1, of periodically processes in terms of their associated multivariate stationary processes are given in section 4. In sections 3 and 5, we study the problem of linear transformation of a harmonizable sequence more closely. Namely, we find spectral conditions under which it is possible to find a mean-convergent series for \( y_n \) in terms of \( x_n \), when \( y_n \) is a linear transformation of \( x_n \). This kind of series expansion is quite
Important in many applications of theory of stochastic processes. The importance of this type of series expansion becomes more transparent when in section 6 we treat the problems of prediction and interpolation of periodically correlated processes as problems in linear transformations of such processes. By using the results of section 5, we find conditions under which it is possible to obtain computable algorithms for the linear predictor and interpolator of periodically correlated processes. Considering the fact that the class of periodically correlated processes is a subclass of harmonizable processes and for this class it is possible to find algorithms for the predictor and interpolator, it is hoped that techniques similar to that of linear transformation would lead to finding algorithms for the predictor and interpolator of a broader subclass of harmonizable processes.

In the following, we introduce some of the notations and concepts which will be used in the sequel.

Let \((\Omega, \mathcal{F}, P)\) be a probability space. \(H = L^2(\Omega, \mathcal{F}, P)\) denotes the Hilbert space of all complex-valued random variables on \(\Omega\) with zero expectation and finite variance. The inner product in \(H\) is defined by

\[
(x, y) = \mathbb{E} xy = \int_{\Omega} x(\omega)\overline{y(\omega)}dP(\omega), \quad x, y \in H.
\]

By a second-order process we mean any sequence \(\{x_n\}_{n \in \mathbb{Z}}\) of elements of \(H\). We use the notation \(x_n\) to denote a second-order process whenever there is no danger of ambiguity.

To every second-order process \(x_n\) we associate the present and past subspaces \(H(x, n)\) defined by,

\[
H(x, n) = \overline{\text{sp}}\{x_k; k \leq n\},
\]

and the terminal subspace defined by

\[
H(x) = \overline{\text{sp}}\{x_n; n \in \mathbb{Z}\},
\]

Where \(\overline{\text{sp}}\) stands for span closure in \(H\).

In simultaneous treatment of two second-order processes the concept of subordination plays a major role. Next, we define the concept of subordination and some related notions for two second-order processes \(x_n\) and \(y_n\).

1.1 Definition. Suppose \(x_n\) and \(y_n\) are second-order processes in
H. We say that:
(i) \( y_n \) is subordinate to \( x_n \) if and only if \( H(y) \subset H(x) \).
(ii) \( y_n \) is strongly subordinate to \( x_n \) if and only if
\[
H(y,n) \subset H(x,n), \text{ for all integers } n.
\]
(iii) \( y_n \) and \( x_n \) are mutually subordinate or equivalent if and only
if \( H(y) = H(x) \).

From this definition of subordination it follows that the
concept of subordination is related to a geometrical property of sub-
spaces in \( H \) associated to the second-order processes. Thus, analyti-
cal characterization of subordination is the problem of finding analytic
conditions for such a geometric property to hold.

2. Subordination of Harmonizable Processes. In this section we study
the problem of subordination of harmonizable processes and its relation
with linear transformation of such processes. First we recall a few
concepts which are essential for this study.

2.1 Definition. A stochastic process \( x_n \) is said to be harmonizable
if \( x_n = \int_0^{2\pi} e^{-in\lambda} Z(d\lambda) \) and

\[
R(m,n) = \text{Ex} \overline{x}_m \overline{x}_n = \int_0^{2\pi} \int_0^{2\pi} e^{-i(m_1-n_1)\lambda_1 + i(m_2-n_2)\lambda_2} (d\lambda_1,d\lambda_2),
\]

where \( Z(\cdot) \) is a random measure defined on the \( \sigma \)-algebra \( B \) of Borel
subsets of \( T = [0,2\pi] \) and for Borel sets \( A, B \), \( \mu(A,B) = E Z(A)Z(B) \)
is such that it extends to a complex measure of bounded variation on
\( T^2 = T \times T \). This complex measure \( \mu \) is referred to as the spectral
measure of the process, c.f. [1].

2.2 The Hilbert space \( L^2(du) \). For \( \phi, \psi \) measurable functions on
\( T \), \( \phi \circ \psi \) denotes the tensor product of \( \phi \) and \( \psi \) i.e.
\[
(\phi \circ \psi)(\lambda_1, \lambda_2) = \phi(\lambda_1)\psi(\lambda_2) \text{ for } \lambda_1, \lambda_2 \in T.
\]

Let \( S \) be the class of all simple functions on \( T \). It is
clear that \( S \) is a linear space and for all \( \phi, \psi \in S \), the double in-
tegral \[
\int_{T^2} \phi \circ \psi \, d\mu = \int_{T^2} \phi(\lambda_1)\psi(\lambda_2) \mu(d\lambda_1, d\lambda_2)
\]
is defined in the
obvious way (\( \mu \) is a measure satisfying (1)).

Two simple functions \( \phi \) and \( \psi \) will be considered identical
if \( \int_{T^2} (\phi - \psi)(\phi - \psi) \, d\mu = 0 \). If we define for \( \phi, \psi \in S \), \( <\phi, \psi> \int_{T^2} \phi \circ \psi \, d\mu \),
then \( (S, <\cdot, \cdot>) \) is an inner product space. In fact, it is obvious
that \( <\phi, \psi> \) has the ordinary bilinear and conjugate symmetric properties
and further \( \langle \phi, \phi \rangle \geq 0 \) (this follows from properties of \( \mu \)), and \( \langle \phi, \phi \rangle = 0 \) if and only if \( \int_\mathcal{T}^2 \phi \ast \overline{\psi} \, d\mu = 0 \). It follows from \( \langle \phi, \phi \rangle \geq 0 \), that we have the Cauchy–Schwarz inequality i.e. \( |\langle \phi, \psi \rangle|^2 \leq \langle \phi, \phi \rangle \langle \psi, \psi \rangle \), for \( \phi, \psi \in \mathcal{S} \).

Let \( \Lambda^2(\mathcal{D}) \) be the completion of \( (\mathcal{S}, \langle \cdot, \cdot \rangle) \) so that it is a Hilbert space with an inner product denoted again by \( \langle \cdot, \cdot \rangle \). Elements in \( \Lambda^2(\mathcal{D}) \) may no longer be functions on \( \mathcal{T} \). A typical element in \( \Lambda^2(\mathcal{D}) \) can be realized as a Cauchy sequence of simple functions. However as a matter of simplicity and for the ease of readers in this manuscript it is convenient that we treat elements in \( \Lambda^2(\mathcal{D}) \) as "formal" functions on \( \mathcal{T} \) and use the improper but suggestive notation \( \int_\mathcal{T}^2 \phi \ast \overline{\psi} \, d\mu \) for the inner product \( \langle \phi, \psi \rangle \) with \( \phi, \psi \in \Lambda^2(\mathcal{D}) \). Of course, \( \int_\mathcal{T}^2 \phi \ast \overline{\psi} \, d\mu = \lim_{n \to \infty} \int_\mathcal{T}^2 \phi_n \ast \overline{\psi}_n \, d\mu \), where \( \phi_n \) and \( \psi_n \) are Cauchy sequences of simple functions from \( \mathcal{S} \) converging to \( \phi \) and \( \psi \) in the norm of \( \Lambda^2(\mathcal{D}) \), respectively.

Let \( \Lambda(\mathcal{D}) = \{ \text{all measurable functions } \phi \text{ on } \mathcal{T} \text{ such that } \int_\mathcal{T}^2 |\phi \ast \overline{\phi}| \, d|\mu| < \infty \text{ and } \int_\mathcal{T}^2 |\phi| \, d|\mu| < \infty \} \), where \( |\mu| \) denotes the total variation measure of \( \mu \) and the double integrals are in the sense of Lebesgue. Clearly each function \( \phi \) in \( \Lambda(\mathcal{D}) \) represents an element \( \phi' \) in \( \Lambda^2(\mathcal{D}) \) in the sense that for all \( \psi \in \mathcal{S} \),

\[
\langle \phi', \psi \rangle = \int_\mathcal{T}^2 \phi'(\lambda_1)\overline{\psi}(\lambda_2) \mu(d\lambda_1, d\lambda_2).
\]

We note that \( \phi' \) is unique, since \( \mathcal{S} \) is dense in \( \Lambda^2(\mathcal{D}) \). We denote \( \phi' \) by \( \phi \) and write \( \phi \in \Lambda^2(\mathcal{D}) \). With this convention and Theorem 1.1 of [4], \( \Lambda(\mathcal{D}) \) is a dense subset of \( \Lambda^2(\mathcal{D}) \) and if \( \phi_1, \phi_2 \in \Lambda(\mathcal{D}) \) with \( \int_\mathcal{T}^2 |\phi_1 \ast \phi_2| \, d|\mu| < \infty \), then \( \langle \phi_1, \phi_2 \rangle \int_\mathcal{T}^2 \phi_1 \ast \overline{\phi_2} \, d\mu \), where the double integral is in the sense of Lebesgue.

For a random measure \( \mathcal{Z}(\cdot) \) as in Definition 2.1, we define \( H(\mathcal{Z}) = \overline{\mathcal{sp}}(\mathcal{Z}(\mathcal{A}) ; \mathcal{A} \in \mathcal{B}) \) in \( \mathcal{H} \). It is shown in [5] that \( \{e_n(\lambda) = e^{-in\lambda} ; n \in \mathbb{Z} \} \) forms a basis in \( \Lambda^2(\mathcal{D}) \), \( H(\mathcal{Z}) = H(\mathcal{X}) = \overline{\mathcal{sp}}(x_n ; n \in \mathbb{Z}) \) and further that there exists an isomorphism between \( \Lambda^2(\mathcal{D}) \) and \( H(\mathcal{Z}) \) defined by \( \phi \ast \int_\mathcal{T} \phi(\lambda) \mathcal{Z}(d\lambda) \), for \( \phi \in \Lambda^2(\mathcal{D}) \).

Careful scrutiny of Kolmogorov and Siraya's work reveal that in problems of subordination a major role is played by cross-correlation of the processes under study. In the following we assume that \( X_n \) and \( Y_n \) are harmonizable processes with covariances;
2.3 Definition. We say that \( x_n \) and \( y_n \) are harmonizably cross-correlated if there exists a complex measure \( F_{xy} \) on \( \mathbb{T}^2 \) such that,

\[
C(m,n) = E x_m \bar{x}_n = \int_{\mathbb{T}^2} e^m \overline{e}_n \, dF_{xx},
\]

\[
B(m,n) = E y_m \bar{y}_n = \int_{\mathbb{T}^2} e^m \overline{e}_n \, dF_{yy}.
\]

2.4 Definition. We say that the harmonizable process \( y_n \) is obtained from \( x_n \) by means of a linear transformation, if there exists a function \( \phi \in \Lambda(dF_{xx}) \) such that

\[
y_n = \int_T e^{-in\lambda} \phi(\lambda) Z(d\lambda), \quad \text{for all integers } n.
\]

2.5 Remark. From Definitions 1.1 and 2.4 it follows easily that when \( y_n \) is obtained from \( x_n \) by means of a linear transformation then \( y_n \) is subordinate to \( x_n \).

By using this remark and the following Theorem which is the analog of Theorem 8.1 [17, p. 36] for harmonizable processes we obtain a sufficient condition for subordination of harmonizable processes which are harmonizably cross-correlated.

2.6 Theorem. Suppose \( x_n \) and \( y_n \) are harmonizable and harmonizably cross-correlated, then \( y_n \) is obtainable from \( x_n \) by means of a linear transformation, if and only if there exists a function \( \phi \in \Lambda(dF_{xx}) \) such that,

\[
dF_{yy} = \phi \otimes \overline{\phi} \, dF_{xx}
\]

(2)

\[
dF_{xy} = \overline{\phi} \, dF_{xx}.
\]

2.7 Remark. By (2) we mean, for any \( A, B \in \mathbb{B} \),

\[
F_{yy}(A,B) = \int_A \int_B \phi(\lambda_1)\overline{\phi(\lambda_2)} \, dF_{xx}(\lambda_1,\lambda_2)
\]

(2')

\[
F_{xy}(A,B) = \int_A \int_B \overline{\phi(\lambda_2)} \, dF_{xx}(\lambda_1,\lambda_2).
\]

Proof of the Theorem. Suppose that there exists \( \phi \in \Lambda(dF_{xx}) \) such that

\[
y_n = \int_T e^{-in\lambda} \phi(\lambda) Z(d\lambda), \quad n \in \mathbb{Z}.
\]
Since $y_n$ is harmonizable, it has its own spectral representation, i.e. there exists a random measure $\xi(\cdot)$, c.f. Definition 2.1, such that

$$y_n = \int_T e^{-in\lambda} \xi(d\lambda).$$

Thus, for all integers $m$ and $n$ we have;

$$\int \int e_m \ast \overline{e_n} \, dF_{xy} = E x_m \overline{y}_n = \int \int e_m \ast \overline{e_n} \, \phi \, dF_{xx}$$

$$\int \int e_m \ast \overline{e_n} \, dF_{yy} = E y_m \overline{y}_n = \int \int (e_m \ast \overline{e_n})(\phi \ast \overline{\phi}) \, dF_{xx}$$

which implies (2).

Conversely, suppose that there exists a $\phi \in A(dF_{xx})$ such that (2) holds. We define $z_n = \int_T e^{-in\lambda} \phi(\lambda)Z(d\lambda)$, then it is easy to check that

$$E x_m \overline{y}_n = E x_m \overline{z}_n$$

$$E y_m \overline{y}_n = E z_m \overline{z}_n.$$ 

Thus, by using a slight extension of Lemma 8.1 [17, p. 35] we get

$$y_n = \int_T e^{-in\lambda} \phi(\lambda)Z(d\lambda),$$

i.e. $y_n$ is obtainable from $x_n$ by means of a linear transformation.

2.8 Theorem. Suppose $x_n$ and $y_n$ are harmonizable and harmonizably cross-correlated. If there exists a function $\phi \in A(dF_{xx})$ such that,

$$dF_{yy} = \phi \ast \phi \, dF_{xx},$$

$$dF_{xy} = \phi \, dF_{xx}$$

Then $y_n$ is subordinate to $x_n$.

It is clear that this theorem is an easy consequence of Theorem 2.6 and Remark 2.5.

2.9 Remark. Theorem 2.8, can also be proved by using Theorem 1 of [20] and the characterization of the reproducing kernel Hilbert space of a harmonizable process, c.f. [13].

2.10 Remark. In Definition 2.1, if $\mu$ is a measure which is concen-
trated on the main diagonal of the square $T^2$, then the corresponding process $x_n$ is stationary. In this case, we can think of $\mu$ as a nonnegative measure on $T$, then it is easy to see that $A^2(\mu)$ and $A(\mu)$ (as defined in section 2.2) are the same as the space of all measureable functions on $T$ which are square integrable with respect to $\mu$, i.e. $A^2(\mu) = A(\mu) = L^2(\mu)$. Thus Theorems 2.6 and 2.8 specialized to the case when $x_n$ and $y_n$ are stationary and stationarily cross-correlated will reduce to Theorem 8.1 of [17, p. 36] and sufficient part of Theorem 9 of [8], respectively.

2.11 A counter example. Here we give a simple example which shows that, unlike the stationary and periodically correlated processes, c.f. Theorem 3.4, the conditions of Theorem 2.8 are not necessary for subordination of harmonizable processes.

Let $\xi$ be a random variable on a probability space with $E\xi = 0$ and $E|\xi|^2 = 1$. Let $f, g \in L^1(T, d\lambda)$ where $f$ is not identically zero. Define the following stochastic processes $x_n = f(n)\xi$ and $y_n = g(n)\xi$, where $f(n) = \frac{1}{2\pi} \int_T e^{-in\lambda} f(\lambda) d\lambda$. It is easy to check that $x_n$ and $y_n$ are harmonizable and harmonizably cross-correlated with $dF_{xx} = f \otimes f dm$, $dF_{yy} = g \otimes g dm$ and $dF_{xy} = f \otimes g dm$, where $m$ is the Lebesgue measure on $T^2$.

For any choice of such functions $f$ and $g$ we have $H(y) = H(x)$, i.e. $y_n$ is subordinate to $x_n$. But, in the following we show that it is possible to choose $f$ and $g$ in such a way that none of the relations in (2) (or (2')) will hold.

Suppose, there exists a $\phi \in A(dF_{xx})$ such that conditions of Theorem 2.8 are satisfied, then for $A = B$ we have

$$\int_A g(\lambda) d\lambda|^2 = \int_A \phi(\lambda)f(\lambda) d\lambda|^2, \quad A \in B. \tag{3}$$

For choice of $A = [0, \pi]$, $g \in L^1(T, d\lambda)$ such that $\int_A g(\lambda) d\lambda \neq 0$ and $f = \chi_{[\pi, 2\pi]}$ we have $\int_A \phi(\lambda)f(\lambda) d\lambda = 0$, which contradicts (3).

2.12 Remark. We note that for stationary processes, the property that $y_n$ is obtainable from $x_n$ by a linear transformation is equivalent to the subordination of $y_n$ to $x_n$ [17, Theorem 8.1] and [8, Theorem 9]. But, this is not the case for harmonizable processes, as counter example 2.11 shows.
3. Linear Transformation of Harmonizable Processes.

Let \( x_n \) and \( y_n \) be harmonizable and harmonizably cross-correlated and \( y_n \) be obtainable from \( x_n \) by means of a linear transformation. Thus, there exists a function \( \phi \in \Lambda(dF_{xx}) \) such that,

\[
y_n = \int_{T} e^{-in\lambda} \phi(\lambda) dZ(\lambda), \quad n \in \mathbb{Z}.
\]

In several important problems in application of harmonizable processes it is desired to write \( y_n \) as a mean-convergent series in the time domain in terms of \( x_n \). In the following, by using some of the ideas developed by one of us in [14,15] we study the possibility of the existence of such a mean-convergent series. Throughout this section we assume that \( \phi \in \Lambda(dF_{xx}) \) is such that \( \phi_k = \int_{T} e^{i\lambda} \phi(\lambda) d\lambda, \quad k \in \mathbb{Z} \), i.e., the \( k \)th Fourier coefficient of \( \phi \) is well-defined. In this direction, we have the following theorem.

3.1 Theorem. Let \( x_n \) and \( y_n \) be harmonizable and harmonizably cross-correlated and \( \phi \in \Lambda(dF_{xx}) \) be such that (4) holds and the \( \phi_k \)'s are well-defined. Then,

(a) \( y_n = \sum_{k=-\infty}^{\infty} \phi_k x_{n-k} \) if and only if the Fourier series of \( \phi \) converges in the norm of \( \Lambda^2(dF_{xx}) \), i.e.

\[
\lim_{n \to \infty} \left\| \phi - \sum_{k=-n}^{n} \phi_k e^{ik\lambda} \right\|_{\Lambda^2(dF_{xx})} = 0.
\]

(b) If \( \phi \in L^2(T,d\lambda) \) and \( dF_{xx} \) is a.c. w.r.t. the Lebesgue measure \( d\lambda \) with the density \( f \in L^{\infty}(T, d\lambda) \), then \( y_n = \sum_{k=-\infty}^{\infty} \phi_k x_{n-k} \), \( n \in \mathbb{Z} \).

Proof. (a) and (b) follow immediately from the isomorphism between \( \Lambda^2(dF_{xx}), H(x) \) and the fact that the Fourier series of an \( L^2(T,d\lambda) \) function converges in the norm of \( L^2(T,d\lambda) \).

3.2 Remark. Theorem 3.1 (a) reduces the problem of writing \( y_n \) in terms of \( x_n \) to the problem of convergence of Fourier series of functions in \( \Lambda(dF_{xx}) \) in the norm of \( \Lambda^2(dF_{xx}) \). Obviously, such a pro-
problem will also arise in the study of stationary processes. In [14] one of us has studied the problem of convergence of Fourier series of functions from the spectral space in the norm of that space and its relation with the measure of the angle between the past and the future of the process. Such a problem has not been studied for harmonizable processes and is an interesting open problem in this field. In section 5 we study this problem for periodically correlated processes which is a subclass of harmonizable processes. We note that because of close connection between periodically correlated processes and multivariate stationary processes, we are able to get more concrete results for the latter class of processes as compared to harmonizable processes, c.f. section 6.

4. Subordination of Periodically Correlated Processes.

A stochastic process $x_n$ is said to be periodically correlated with period $q$ if the function $R(m,n) = \langle x_{m+n}, x_m \rangle$ is periodic in $m$ with period $q$ (we note that when $q = 1$ the process is stationary). Since $R(m,n)$ is periodic in $m$ with period $q$, one can write

$$R(m,n) = \sum_{k=1}^{q} R_k(n) \exp \left(\frac{2\pi i km}{q}\right).$$

For convenience we extend the definition of these functions $R_k(n), k = 1,2, \ldots, q$ to all integers by $R_k(n) = R_{k+q}(n)$.

It is shown in [6] that each $R_k(n), n \in \mathbb{Z}$, has the representation,

$$R_k(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\lambda} dF_k(\lambda),$$

where $F_k(\cdot)$ is a complex-valued measure on $[-\pi, \pi]$. Let $F(\cdot)$ be the $q \times q$ matrix-valued measure defined on intervals $(\lambda_1, \lambda_2)$ by

$$F(\lambda_1, \lambda_2) = \left[ F_{k-j}(\frac{\lambda_1 + 2\pi j}{q}, \frac{\lambda_2 + 2\pi j}{q}) \right]_{j,k=0}^{q-1}$$

It is proved in [6] that $F(\cdot)$ is a nonnegative definite hermitian matrix-valued measure. It is also shown in [6] that,

$$R(m,n) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-i(m+n)\lambda_1 + im\lambda_2} dF(\lambda_1, \lambda_2),$$

where the spectral measure $F(\cdot, \cdot)$ is given by

$$F(A,B) = \frac{1}{q} \sum_{k=-q+1}^{q-1} \int_{A\cap(B-\frac{2\pi k}{q})} dF_k(\lambda).$$
(B-a is the set of all b-a with b \in B). In other words the spectral measure \( F(\cdot, \cdot) \) in (6) is concentrated on \( 2q-1 \) straight line segments

\[
\lambda_1 - \lambda_2 = \frac{2\pi k}{q}, \quad k = -q+1, \ldots, q-1,
\]

contained inside the square \( 0 \leq \lambda_1, \lambda_2 \leq 2\pi \), and the measures \( F_k(\cdot) \) give the mass of \( F(\cdot, \cdot) \) on these lines according to (7). Representations (1) and (6) and formula (7) show that any periodically correlated process is harmonizable. Thus, periodically correlated process is a subclass of harmonizable processes.

With any \( H \)-valued process \( x_n \) we associate the \( H^q \)-valued process \( X_n \) whose \( i \)-th coordinate is given by \( x_{nq + i} \), \( i = 0, 1, 2, \ldots, q-1 \). This correspondence establishes a one-to-one linear transformation \( S_q \) from the \( \mathcal{H} \)-valued processes onto the \( H^q \)-valued processes and we have

\[
X_n = (S_q x)_n = \begin{bmatrix}
x_{nq} \\
x_{nq + 1} \\
\vdots \\
x_{nq + q-1}
\end{bmatrix}
\]

We will simply write \( S \) instead of \( S_q \). It can easily be verified that \( X_n \) is periodically correlated with period \( q \) if and only if \( X_n = (Sx)_n \) is a \( q \)-variate stationary stochastic process. (For the theory of multivariate stationary stochastic processes, see Masani [10]). As we know this associated \( q \)-variate stationary stochastic process has a spectral measure \( F \) which is a \( q \times q \) nonnegative definite matrix-valued measure such that,

\[
(X_n, X_0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\lambda} dF(\lambda).
\]

The following theorem \[6,11\] gives the relation between the measures \( F \) and \( F \).

4.1 Theorem. With notations as above we have,

\[
dF(\lambda) = q \, U^*(\lambda) dF(\lambda) U(\lambda);
\]

in the sense that given any set \( A \),

\[
F(A) = \int_A q \, U^*(\lambda) dF(\lambda) U(\lambda).
\]
Here $U$ is a unitary matrix-valued function whose $(j,k)$th entry is
\[ q^{-\frac{1}{2}j} \exp\left[ \frac{2\pi i j k + i k \lambda}{q} \right]. \]

In the following by using this one-to-one correspondence between periodically correlated processes and multivariate stationary processes, we find necessary and sufficient conditions for subordination, mutual subordination and a necessary condition for strong subordination of periodically correlated processes in terms of their associated stationary processes. Thus, spectral necessary and sufficient conditions for subordination and mutual subordination of periodically correlated processes can be obtained by using the results of section 8 [10], c.f. Remark 5.3.

Throughout this section we assume that $x_n$ and $y_n$ are periodically correlated processes with period $q$ and that they are periodically cross-correlated, i.e. the function $\hat{R}_{xy}(m,n) = (x_{m+n}, y_m)$ is periodic in $m$ with period $q$. (We will simply say that $x_n$ and $y_n$ are periodically cross-correlated with period $q$).

4.2 Remark. If $x_n$ and $y_n$ are periodically cross-correlated with period $q$, then $[x_n, y_n]$, $n \in \mathbb{Z}$, is a two-dimensional periodically correlated process. Thus, $R_{xy}(\cdot, \cdot)$ has a spectral representation, c.f. [6], similar to the spectral representation of covariance of $x_n$.

4.3 Lemma. If $x_n$ and $y_n$ are periodically cross-correlated with period $q$ and $X_n, Y_n$ are their associated q-variate stationary processes. Then $X_n$ and $Y_n$ are stationarily cross-correlated.

Proof. It is easy to check that, for all integers $m,n$;
\[
(X_m, Y_n) = \left( E x_{mq+i} \bar{y}_{nq+j} \right)^{q-1} = \left( E x_{(m-n)q+i} \bar{y}_{j} \right)^{q-1}_{i,j=0},
\]
which depends on $m-n$ alone.

For $x_n$ a periodically correlated process and $X_n$ its associated q-variate stationary process we have for all integers $n$,
\[
H(X, n) = \text{sp}\{x_{mq+i} ; m \leq n, 0 \leq i \leq q-1\} = \text{sp}\{x_{k} ; k \leq nq+q-1\} = H(x, nq+q-1).
\]

By letting $n \to \infty$, we get the following important equality,
Suppose \( x_n \) and \( y_n \) are periodically cross-correlated processes with period \( q \) and \( X_n, Y_n \) are their associated stationary processes, then

(i) \( y_n \) is subordinate to \( x_n \) if and only if \( Y_n \) is subordinate to \( X_n \).

(ii) \( y_n \) and \( x_n \) are mutually subordinate if and only if \( Y_n \) and \( X_n \) are mutually subordinate.

Proof. (1) and (ii) are obvious because of Lemma 4.3, relation (9) and (10; relation 2.13).

4.5 Remark. If \( y_n \) is strongly subordinate to \( x_n \), then from (8) it follows that \( Y_n \) is also strongly subordinate to \( X_n \). But, the converse is not necessarily true. For an example, let \( \xi_n \) be a periodically correlated process with period \( q = 2 \) and \( \xi_n = \mathbf{sp}(\xi_k; k \leq n-1) \).

Define \( x_n \) and \( y_n \) by \( x_{2n} = \xi_{2n-1} \), \( x_{2n+1} = \xi_{2n} \), \( y_{2n} = \xi_{2n} \) and \( y_{2n+1} = \xi_{2n-1} \), then \( x_n \) and \( y_n \) are periodically cross-correlated of period 2, and it is clear that \( Y_n \) is strongly subordinate to \( X_n \), but \( H(y,2n) \supset H(x,2n) \), i.e. \( y_n \) is not strongly subordinate to \( x_n \).

5. Linear Transformation of Periodically Correlated Processes.

In this section, we use extensively the fact that there is a one-to-one correspondence between periodically correlated processes with period \( q \) and \( q \)-variate stationary processes. We state most of our results in terms of \( q \)-variate processes, the corresponding results for periodically correlated processes can be obtained by rewriting the appropriate expressions in terms of periodically correlated processes, c.f. Remark 5.2. In this section we assume that \( x_n \) and \( y_n \) are periodically correlated and cross-correlated with period \( q \) and that \( y_n \) is obtainable from \( x_n \) by means of a linear transformation, that is

\[
y_n = J_{2n} e^{-i n \lambda} \Phi(\lambda) dZ(\lambda), \quad n \in \mathbb{Z}.
\]

Where \( Z(\cdot) \) is the (random) spectral measure of the \( q \)-variate stationary process \( X_n \) (associated to \( x_n \)) and \( \Phi \) is a \( q \times q \) matrix-valued function in \( L^2(F) \), c.f. [10]. Here \( F \) is the spectral measure of \( X_n \) and \( L^2(F) \) is the spectral space corresponding to \( F \).
As a measure of angle between past and future of the process $X_n$, we define $\rho(F) = \sup \{(P,F)\},$ where $P$ and $F$ vary over the unit balls of $H(X,0) = \overline{sp}\{X_k; k \leq 0\}$ and $H^\infty(X) = \overline{sp}\{X_k; k \geq 1\}$ in $H^\infty$, which correspond to the past and future of the process $X_n$. It is clear that $\rho(F) \leq 1$. It is said that the past and future of the process $X_n$ are at positive angle if $\rho(F) < 1$. This measure of angle between the past and future of the process $X_n$ plays an important role in writing the expression (10) as a mean-convergent series in terms of $X_n, n \in \mathbb{Z}$. In the following we assume that $F$ is such that for every $\phi \in L^2(F’), its k-th Fourier coefficient is well-defined i.e. for $k \in \mathbb{Z},$

$$\phi_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\lambda} \phi(\lambda) d\lambda$$

is well-defined. If the measure $F$ is a.c. w.r.t. the 0 Lebesgue measure we denote its density by $f$. When there is danger of confusion the letters $f,F,Z,$ etc. corresponding to a process $X_n$ will be denoted by $f_X,F_X,Z_X,$ etc. With notations as above we have the following theorem.

5.1 Theorem. Let $Y_n = \int_{-\pi}^{\pi} e^{-in\lambda} \phi(\lambda) dZ_X(\lambda), n \in \mathbb{Z}$ for some $\phi \in L^2(F_X)$. Then,

a) $Y_n = \sum_{k=-\infty}^{\infty} \phi_k X_{n-k}$ if and only if the Fourier series of $\phi$ converges in $L^2(F_X)$.

b) $Y_n = \sum_{k=-\infty}^{\infty} \phi_k X_{n-k}$ if $\rho(F_X) < 1$.

c) $Y_n = \sum_{k=-\infty}^{\infty} \phi_k X_{n-k}$ if $\phi \in L^2_{q \times q}(d\lambda)$ and $f_X \in L^\infty_{q \times q}(d\lambda)(L^2_{q \times q}(d\lambda)$, where $L^\infty_{q \times q}(d\lambda)$ stands for the usual class of $q \times q$ matrix-valued functions c.f. [10]).

Proof. See [15], Theorems 6.1, 7.3.

5.2 Remark. If $X_n$ and $Y_n$ are periodically correlated and cross-correlated with period $q$ and $Y_n$ is obtainable from $X_n$ by means of a linear transformation, then it follows that under the conditions of Theorem 5.1(a), (b) or (c) we have

$$Y_{nq+i} = \sum_{k=-\infty}^{\infty} \sum_{j=0}^{q-1} \phi^i_k X_{(n-k)q+j},$$

$i = 0,1,2,\ldots,q-1$, where $\phi^i_k$ is the $(ij)$-th entry of $\phi_k$.

5.3 Remark. In theorems 4.4 and 5.1 almost all the conditions are stated in terms of the spectral measure of the process $X_n$ i.e. $F_X$. 
It is of interest to know whether these conditions can also be stated in terms of \( F_X \), c.f. (5). In the case of Theorem 5.1(b), the above problem reduces to the following interesting question:

Let \( X_n = \int_{-\pi}^{\pi} e^{-in\lambda} d\mathcal{Z}(\lambda) \) be a \( q \)-variate stationary process with the spectral measure \( F_X \) and \( U(\cdot) \) be a \( qxq \) unitary matrix-valued function. Define the \( q \)-variate stationary process \( Y_n \) by \( Y_n = \int_{-\pi}^{\pi} e^{-in\lambda} u(\lambda)d\mathcal{Z}(\lambda), n \in \mathbb{Z} \), and the spectral measure \( F_y \). Here is our question: Is it true that \( \rho(F_X) = \rho(F_y) \)? A positive answer to this question will be a major contribution toward solving the problem of strong mixing of vector-valued Gaussian stationary processes, c.f. [14]. We note that the motivation for the above question comes from a simple geometrical fact that, the angle between two subspaces in a Hilbert space is preserved under a rotation.

6. Algorithms for the Linear Predictor and Interpolator for Periodic Case.

The problem of finding algorithms for determining the best linear predictor and interpolator is important in most applications of the theory of stochastic processes.

In this section by using Theorem 5.1 we find such algorithms for periodically correlated processes. The problem of prediction of the periodically correlated processes has been studied in detail in [11] and an algorithm for finding the linear predictor is also provided under a strong condition on the spectrum of the process. Here, we show that the algorithm of [11] is also available under a different set of conditions on the spectrum of the process. Also, we show that, essentially under the same set of conditions it is possible to find an algorithm for the linear interpolator of the periodically correlated processes.

Let \( x_n \) be a periodically correlated process with period \( q \). For any given time \( n \), the best linear predictor of \( x_n \), in the least square sense, with respect to the past of \( x_n \) process up to time \( n-1 \) is simply \( (x_n | H(x,n-1)) \) for all integers \( n \), where \((x_n | H(x,n-1))\) denotes the orthogonal projection of \( x_n \) on the subspace \( H(x,n-1) \) of \( H(x) \). But, because of periodicity it is enough to compute these predictors just for integers \( n = 0,1, ..., q-1 \). Let \( X_n \) be the associated \( q \)-variate stationary process of \( x_n \) with the spectral measure \( F_X \).

From now on we assume that \( X_n \) is purely-nondeterministic i.e. \( F_X \) is a.c. w.r.t. the Lebesgue measure with density \( f_X \) such that \( \log \det f_X \) is in \( L^1(d\lambda) \). Under the additional condition that the \( f_X^{-1} \) has summable entries \( f_X \) admits a factorization of the form \( f_X(\lambda) = \phi(\lambda)\phi^*(\lambda) \),
where the optimal factor $\Phi$ and its reciprocal $\Phi^{-1}$ have square integrable entries and have only nonnegative Fourier coefficients:

$$\Phi(\lambda) = \sum_{k=0}^{\infty} C_k e^{ik\lambda}, \quad \Phi^{-1}(\lambda) = \sum_{k=0}^{\infty} D_k e^{ik\lambda} \quad (c.f. \ [10]).$$

It is shown in [11, pp. 176-178] that the linear predictor of $X_n$ plays a crucial role in finding the linear predictor of $X_0$. In the next theorem, by using Theorem 5.1, we find conditions on $f_X$ such that $\hat{X}_0$, the linear predictor of $X_0$, has a mean-convergent series expansion in terms of $(X_k; k \leq -1)$. It is known [10] that

$$\hat{X}_0 = \int_{-\pi}^{\pi} \Phi_1(\lambda) dZ(\lambda), \quad \text{where} \quad \Phi_1(\lambda) = [e^{-i\lambda}\Phi(\lambda)]_+ \Phi^{-1}(\lambda)$$

is the isomorph of $\hat{X}_0$ in $L^2(f_X)$ under the Kolomogorov's isomorphism and

$$\left\{ \sum_{k=-\infty}^{\infty} \Phi_k e^{ik\lambda} \right\}_+ = \sum_{k=0}^{\infty} \Phi_k e^{ik\lambda}.$$ 

With notation as above the following theorem is immediate from Theorem 5.1 and [10].

6.1 Theorem. Let $f_X^{-1}$ have summable entries. Then (a) $\hat{X}_0 = \sum_{k=1}^{\infty} E_k X_{-k}$ if and only if the Fourier series of $\Phi_1$ converges in the norm of $L^2(f_X)$, where $E_k = \sum_{n=1}^{k} C_n D_{k-n}$.

b) $\hat{X}_0 = \sum_{k=1}^{\infty} E_k X_{-k}$ if $\rho(f_X) < 1$.

c) $\hat{X}_0 = \sum_{k=1}^{\infty} E_k X_{-k}$ if $f_X \in L_{qXq}^\infty(\text{d}\lambda)$.

6.2 Remark. (a) The algorithm for finding the linear predictor of $X_n$, $n = 0,1,\ldots,q-1$ based on $(x_k; k \leq n-1)$ $n = 0,1,\ldots,q-1$ respectively, can be obtained by using remark 5.2, Theorem 6.1 and the technique of [11, pp. 177-178]. (b) We note that in [11] the algorithm for finding $\hat{X}_0$ is given only under the condition (c) of Theorem 6.1.

To interpolate the missing values of $x_k$, $k \in T$, based on the observed values of $x_k$, $k \in T = Z \setminus T$, where $T$ is a finite subset of the set of all integers $Z$, the problem can be reduced to a similar problem in terms of the associated q-variate process $X_n$. Here, for the sake of simplicity we assume that $T = \{0,1,2,\ldots,q-1\}$. We note that the important case of $T = \{n\}, n = 0,1,2,\ldots,q-1$ can be handled by using our Theorem 6.3 and the technique of [11, p. 177-178]. The other cases of $T$ can be treated similarly by using the results of [17, p. 101].
Since the missing values \( x_n, n = 0, 1, 2, \ldots, q-1 \), correspond to \( X_0 = [x_0, x_1, \ldots, x_{q-1}] \), we have to interpolate \( X_0 \) based on \( X_n, n \neq 0 \), in the least square sense, i.e. we have to find \( \hat{X}_0 = (X_0 | H(X, n \neq 0)) \), where \( H(X, n \neq 0) = \overline{sp}(X_n; n \neq 0) \) in \( H^q \).

From now on, we assume that the process \( X_n \) is minimal i.e. the entries of both \( f_x \) and \( f_x^{-1} \) are summable. Under this assumption, it is shown in [10] and [17] that the isomorph of \( \hat{X}_0 \) in \( L^2(f_x) \) is the function \( I - C_0^{-1} f_x^{-1} \), thus we have; \( \hat{X}_0 = \int_{-\pi}^{\pi} (I - C_0^{-1} f_x^{-1}(\lambda))dZ(\lambda) \), where \( I \) is the \( q \times q \) identity matrix and \( C_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\lambda} f_x^{-1}(\lambda)d\lambda \), i.e. the \( k \)-th Fourier coefficient of the inverse of \( f_x \). Now in view of Theorem 5.1 and Remark 6.2 the following Theorem may be used to obtain an algorithm for the linear interpolator of a periodically correlated process.

6.3 Theorem. Let \( x_n \) be a minimal periodically correlated process and \( X_n \) be its associated \( q \)-variate stationary process. Then,

(a) \( \hat{X}_0 = \sum_{k \neq 0} (-C_0^{-1}C_k)X_k \) if and only if the Fourier series of \( f_x^{-1} \) converges in the norm of \( L^2(f_x) \).

(b) \( \hat{X}_0 = \sum_{k \neq 0} (-C_0^{-1}C_k)X_k \) if \( \rho(f_x) < 1 \).

(c) \( \hat{X}_0 = \sum_{k \neq 0} (-C_0^{-1}C_k)X_k \) if \( f_x \in L^{\infty} (d\lambda) \) and \( f_x^{-1} \in L^2_{q \times q} (d\lambda) \).

We note that the sufficient conditions in Theorems 6.1(c) and 6.3(c) can also be written in terms of the spectral density matrix of \( F_x \), c.f. Remark 5.3.

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PROPERTIES OF SEMISTABLE PROBABILITY MEASURES ON $\mathbb{R}^m$

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1. INTRODUCTION AND PRELIMINARIES

Let $0 < r < 1$ and let $B$ be a real separable Banach space. A Borel probability measure $\mu$ on $B$ is called $r$-semistable if there exist $x_n \in B$, $a_n > 0$, a sequence $\{k_n\}$ of positive integers and a Borel $\sigma$-measure $\nu$ on $B$ such that

$$k_n^{-1} = r \quad \text{and} \quad a_n \cdot *k_n * \delta_{x_n} \ast \nu \ast \mu,$$

(1.1)
as $n \to \infty$, where $*$ and $\ast$ denote, respectively, the convolution of measures and the usual weak convergence of measures; and, for a given number $a \neq 0$ and a measure $\rho$, $a \cdot \rho = \rho \circ \tau_a^{-1}$, where $\tau_a(x) = ax$. It is now known that $\mu$ is $r$-semistable if and only if $\mu$ is infinitely divisible (i.d.) and

$$\mu^{*n} = r^{n/\alpha} \cdot \mu \ast \delta_{x_n},$$

(1.2)
for every $n = \pm 1, \pm 2, \ldots$, where $\mu^t$ denotes the $t$-th power of $\mu$, $0 < \alpha < 2$, and is uniquely determined by $\mu$; $\alpha$ is referred to as the index of $\mu$. If $x_n$ in (1.1) or equivalently in (1.2) is equal to 0, for every $n$, then $\mu$ is called strictly $r$-semistable. We refer the readers to [3] for the above and another characterization of $r$-semistable $\mu$-measures on locally convex spaces.
For a given $0 < r < 1$ and $0 < \alpha < 2$, throughout, we shall use the notation $r$-SS($\alpha$) (respectively, $S(\alpha)$) for the phrase "$r$-semistable index $\alpha$" (respectively, for "stable index $\alpha$".) If $X$ is a random variable (r.v.) with values in a separable Banach space $B$, then $\mathcal{L}_X$ will denote the law of $X$ in $B$. In a Banach space $B$, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will, respectively, denote its norm and the natural duality between $B$ and $B^*$, the topological dual of $B$. For a given $0 < r < 1$ and $0 < \alpha < 2$, throughout $\Delta$ will denote the set $\{x \in B : r^{1/\alpha} \|x\| \leq 1\}$. If $\mu$ is a p. measure on a Banach space $B$, then $\hat{\mu}$ will denote the characteristic (ch.) function of $\mu$. If $T$ is a metric space, then $\mathcal{B}(T)$ will denote the class of its Borel sets.

The main purpose of this paper is three fold: First we obtain characterizations of rotationally invariant $r$-SS($\alpha$) p. measures on $R^m$, the $m$-Euclidean space; second we obtain ch. functions of sub-semistable and other related p. measures on $R^m$, and third we obtain necessary and sufficient condition for linearity of regression for i.d. r.v.'s. In addition, we provide examples of two symmetric $r$-SS($\alpha$), $1 < \alpha < 2$, as well as non-symmetric $S(\alpha)$, r.v.'s $X_0$, $X_1$ where the regression of $X_0$ on $X_1$ is not linear.

We close this section by quoting a theorem from [11] which will be needed for several proofs.

**THEOREM 1.1.** Let $0 < r < 1$, $0 < \alpha < 2$, and let $\mu$ be a symmetric $r$-SS($\alpha$) p. measure on $R^m$, with (symmetric) Levy measure $F$, then for every $y \in R^m$,

$$
\hat{\mu}(y) = \exp\left\{-\int_{\Delta} |\langle x, y \rangle|^\alpha k_\alpha(\langle x, y \rangle) \, dx \right\}, \tag{1.3}
$$

where $k_\alpha(t) = |t|^{-\alpha} \sum_{n=0}^\infty r^{-n}(1 - \cos r^{n/\alpha} t)$, $t \neq 0$, $k_\alpha(0) = 0$ and $F = F/\Delta$. Conversely, if $\Gamma$ is any symmetric measure on a Borel subset $T$ of $R^m$ satisfying

$$
\int_T \|x\|^\alpha \, dx < \infty, \tag{1.4}
$$

then $\hat{\mu}$ given by (1.3) with $\Delta$ replaced by $T$, is the ch. function of a symmetric $r$-SS($\alpha$) p. measure. If $T = \Delta$ (in which case (1.4) is equivalent to
Γ(Δ) < ∞) then μ determines uniquely the measure Γ (Γ is called the spectral measure of μ). Further, the corresponding Lévy measure F of μ is given by

\[ F(A) = \sum_{n=0}^{\infty} r^n \Gamma(r^{n/\alpha} A \cap T), \]

for every Borel set A. (Here and in the rest of the paper \( \sum_{n=-\infty}^{\infty} \) stands for \( \sum_{n=0}^{\infty} \)).

2. CHARACTERISTIC FUNCTIONS OF ROTATIONALLY INVARIANT AND RELATED P. MEASURES ON \( \mathbb{R}^m \)

In this section, we give two characterizations, in terms of ch. functions, of rotationally invariant r-SS(α) p. measures; we also obtain ch. functions of two other related r-SS(α) p. measures. We begin with some preliminaries: Let μ be an i.d. p. measure on \( \mathbb{R}^m \) with (symmetric) Lévy measure F. It is known and not hard to show that μ is rotationally invariant if F is rotationally invariant:

\[ F = (\lambda \times \sigma_m) \circ \tau^{-1}, \]

where \( \sigma_m \) is the surface measure on the unit sphere \( \partial S \) of \( \mathbb{R}^m \), \( \lambda \) is the Borel measure on \([0, \infty)\) given by \( \lambda(A) = [\sigma_m(\partial S)]^{-1} F(x \in \mathbb{R}^m: \|x\| \in A) \), and \( \tau \) is the map: \([0, \infty) \times \partial S \rightarrow \mathbb{R}^m \) defined by \( \tau(t, x) = tx \). Using these facts one can easily prove the following characterization (Theorem 2.1(a)) of i.d. rotationally invariant p. measures on \( \mathbb{R}^m \); simple proof is included for completeness. Throughout the paper we shall reserve the notations \( \sigma_m \) and \( s_0 \) for the surface measure on the unit sphere \( \partial S \) of \( \mathbb{R}^m \) and for \( \sigma_m(\partial S) \), respectively. In the following theorem \( J_s \) denotes the Bessel function of order \( s \) and the integer \( m \) is assumed to be larger than one.

THEOREM 2.1(a). Let μ be an i.d. rotationally invariant p. measure on \( \mathbb{R}^m \), then, for every \( y \in \mathbb{R}^m \),

\[
\hat{\mu}(y) = \exp \left[ -\int_0^\infty \left( \frac{t\|y\|^2}{2} - \frac{m-2}{2} s_0^2 \right) \lambda(dt) \right],
\]

(2.2)
where $\lambda$ is a unique measure on $(0, \infty)$ satisfying
\[ \int_0^\infty \min(1, t^2) \lambda(dt) < \infty. \]  

Conversely, for any measure $\lambda$ on $(0, \infty)$ satisfying (2.3), the function $\hat{\mu}$ given by (2.2) is the ch. function of a rotationally invariant i.d. p. measure on $\mathbb{R}^m$.

(b) Let $0 < r < 1$, $0 < \alpha < 2$ and let $\mu$ be a rotationally invariant $r$-SS($\alpha$) p. measure on $\mathbb{R}^m$, then, for every $y \in \mathbb{R}^m$,
\[ \hat{\mu}(y) = \exp\left[ \frac{1}{(r^{1/\alpha}, 1]} \int \left\{ \sum_{n=1}^{m-2} \left( \frac{r^{-n} (\frac{2}{n} - \frac{m-2}{2})}{(t r^{n/\|y\|^2})} \right) \lambda(dt) \right\}, \right. \]  

where $\lambda$ is a finite measure on $(r^{1/\alpha}, 1]$ and is uniquely determined by $\mu$.

Conversely, for any finite measure $\lambda$ on $(r^{1/\alpha}, 1]$, the function $\hat{\mu}$ given by (2.4) is the ch. function of a rotationally invariant $r$-SS($\alpha$) p. measure on $\mathbb{R}^m$.

(c) Let $0 < r < 1$, $0 < \alpha < 2$ and let $\mu$ be a rotationally invariant $r$-SS($\alpha$) p. measure on $\mathbb{R}^m$, then
\[ \hat{\mu}(y) = \exp\left[ -\|y\|^\alpha \int_{\{r^{1/\alpha} < |t| \leq 1\}} |t|^\alpha k_\alpha(t \|y\|) \lambda(dt) \right], \]  

for every $y \in \mathbb{R}^m$, where $\lambda$ is a unique finite symmetric measure on $\{r^{1/\alpha} < |t| \leq 1\}$ satisfying
\[ \lambda(A) = \sum_{n=0}^\infty r^n \Gamma\{x \in \Delta : \langle x, u \rangle \in r^{n/\alpha} A\}, \]  

for every Borel set $A$ of $\{r^{1/\alpha} < |t| \leq 1\}$; in (2.6) $\Gamma$ is the (rotationally invariant) spectral measure of $\mu$ and $u$ is any unit vector in $\mathbb{R}^m$.

Conversely, if $\lambda$ is a finite symmetric measure on $\{r^{1/\alpha} \leq |t| \leq 1\}$ given by (2.6) for a finite rotationally invariant measure $\Gamma$ on $\Delta$ and a unit vector $u$, ...
then \( \hat{\mu} \) given by (2.5) is the ch. function of a rotationally variant r-SS(\( \alpha \)) p. measure on \( \mathbb{R}^m \).

**Proof of (a):** Let \( F \) be the Lévy measure of \( \mu \) then, by (2.1),
\[
F = (\lambda \times \sigma_m) \circ \tau^{-1}.
\]
Hence
\[
\hat{\mu}(y) = \exp\left(\int_{\mathbb{R}^m} \left[\cos(x, y) - 1\right] dF(x)\right)
\]
\[
= \exp\left(\int_{\mathbb{S}} \int_0^\infty \left[\cos(t, y) - 1\right] \lambda(dt) \sigma_m(d\omega)\right)
\]
\[
= \exp\left(\int_0^\infty \left[\int_{\mathbb{S}} \cos(t, y) \sigma_m(d\omega) - s_0\right] \lambda(dt)\right)
\]
\[
= \exp\left(\int_0^\infty \left(-\frac{m-2}{2m} \frac{m-2}{2} - s_0\right) \lambda(dt)\right),
\]
(2.7)
(see, e.g., [13, p. 248]). That \( \lambda \) satisfies (2.3) follows from the fact that \( F \) is a Lévy measure and from the definition of \( \lambda \). If \( \lambda' \) is another measure satisfying (2.3) such that \( \hat{\mu}(y) \) equals the expression on the right side of (2.2) with \( \lambda \) replaced by \( \lambda' \), then, by reversing the steps in the derivation of (2.7), one obtains, using uniqueness of the Lévy measure, that \((\lambda \times \sigma_m) \circ \tau^{-1} = (\lambda' \times \sigma_m) \circ \tau^{-1}\); which implies \( \lambda = \lambda' \). Similar arguments yield a proof of the converse part.

**Proof of (b):** The proof of this part is similar to that of part (a); the only main difference here is that one uses the representation of the ch. function of an r-SS(\( \alpha \)) p. measure given in Theorem 1.1 instead of the representation of the ch. function of an i.d. p. measure used in part (a).

**Proof of (c):** Let \( \mu \) be an r-SS(\( \alpha \)) rotationally invariant p. measure, then \( \Gamma \), the spectral measure of \( \mu \), is also rotationally invariant; and, from Theorem 1.1,
\[
\hat{\mu}(y) = \exp\left(-\int_{\Delta} |\langle x, y \rangle|^{\alpha} k_\alpha(\langle x, y \rangle) \Gamma(dx)\right),
\]
(2.8)
for every \( y \in \mathbb{R}^m \). Let \( u \) be a fixed unit vector of \( \mathbb{R}^m \), define the measure on \( \mathbb{R} \) by

\[
\nu(A) = \Gamma \{ x \in \Delta; \quad \langle x, u \rangle \in A \}
\]

for every Borel set of \( A \). Using rotational invariance of \( \Gamma \), it follows that \( \nu \) does not depend on \( u \) and that for any \( y \in \mathbb{R}^m \), \( y \neq 0 \),

\[
\Gamma \{ x; \quad \langle x, y \rangle \in A \} = \nu \left( \frac{A}{\|y\|} \right);
\]

using this, the facts that \( \nu \{ |t| > 1 \} = 0 \) and that \( \nu \) is symmetric, it follows from (2.8) that, for every \( y \in \mathbb{R}^m \),

\[
\hat{\mu}(y) = \exp \left\{ \int_{|t| \leq 1} |t|^{\alpha} \|y\|^\alpha k_\alpha(t\|y\|) \, \nu(dt) \right\}
\]

\[
= \exp \left\{ 2 \int_{U} \frac{n+1}{r^{\alpha}} \left( r^{\alpha}, r^{n/\alpha} \right) \left| t \right|^{\alpha} \|y\|^\alpha k_\alpha(t\|y\|) \, \nu(dt) \right\}
\]

\[
= \exp \left\{ \|y\|^{\alpha} \int_{(r^{1/\alpha},1]} \left| t \right|^{\alpha} k_\alpha(t\|y\|) \lambda(dt) \right\},
\]

where \( \lambda(A) = \sum_{\alpha n = 0}^{\infty} r^n \nu(r^{n/\alpha} A) \), for every Borel set \( A \). Since \( \lambda \) is symmetric and finite (as \( \sum_{\alpha n = 0}^{\infty} r^n < \infty \) and \( \Gamma(\Delta) < \infty \)), it follows that \( \hat{\mu} \) has the desired representation. (We note that if \( m = 1 \), then \( \Gamma = \nu = \lambda \), and the representation of \( \hat{\mu} \) reduces to the one given in Theorem 1.1. In the case \( m > 1 \), the complicated nature of the measure \( \lambda \) makes the representation of \( \hat{\mu} \) remarkably different as compared to the case \( m = 1 \), see also Remark (2.2) for more on this point). If \( \lambda_1 \) is another finite symmetric measure on \( \{ r^{1/\alpha} \leq |t| \leq 1 \} \) such that (2.5) holds with \( \lambda \) replaced by \( \lambda_1 \), then taking \( y = (x,0,...,0) \) one obtains
for all \( x \). This, along with Theorem 1.1 yield \( A = A_1 \).

Conversely, if \( A \) is given by (2.6) and \( u \) by (2.5), then by reversing the steps used to obtain the representation of \( \hat{\mu} \) above one notes that \( \hat{\mu}(y) \) is the right side of (2.8), which along with the fact that \( \Gamma \) is rotationally invariant shows, via Theorem 1.1, that \( \hat{\mu} \) is the ch. function of a rotationally invariant \( r\text{-SS}(\alpha) \) p. measure. This completes the proof of the theorem.

**REMARK 2.2.** With the analogy of the stable case, one may wonder if \( \phi(y) = \exp\{-\|y\|^{\alpha} k_\alpha(\|y\|)\} \) is the ch. function of a rotationally invariant \( r\text{-SS}(\alpha) \) p. measure on \( \mathbb{R}^m \). As is clear from Theorem 1.1, for the case \( m = 1 \), this is indeed the ch. function of a symmetric \( r\text{-SS}(\alpha) \) p. measure; but surprisingly for \( m > 1 \), this is not even a ch. function: For if it is indeed a ch. function, then using the fact \( \phi^n(y) = \phi(\frac{y}{\sqrt{n}}) \), \( \phi^{\frac{m}{n}} \) is also a ch. function, and hence by a Theorem of [2], \( \phi^n \) is the ch. function for all \( n \); showing \( \phi \) is the ch. function of an i.d. and hence rotationally invariant \( r\text{-SS}(\alpha) \) p. measure on \( \mathbb{R}^m \). Thus, from Theorem 2.1(c),

\[
\|y\|^{\alpha} k_\alpha(\|y\|) = \int_{\{|r^{1/\alpha} < |t| \leq 1\}} |t|^{\alpha} \|y\|^{\alpha} k_\alpha(t\|y\|) \lambda(dt),
\]

for a unique symmetric Borel measure \( \lambda \) on \( \{|r^{1/\alpha} < |t| \leq 1\} \); showing

\( \lambda = 1/2 \delta_1 + 1/2 \delta_{-1} \). Thus, according to (2.6), \( \sum_{n=0}^{\infty} \Gamma\{x \in A : \langle x, u \rangle \in \frac{r^n}{\sqrt{n}} A\} = 1/2 \delta_1(A) + 1/2 \delta_{-1}(A) \), where \( \Gamma \) is the corresponding spectral measure. This shows \( \Gamma\{x \in A : \langle x, u \rangle \notin \{\frac{r^n}{\sqrt{n}}, -\frac{r^n}{\sqrt{n}}\}\} = 0 \), for every \( n \); i.e.,

\( \Gamma\{x : \langle x, u \rangle \in \{\frac{r^n}{\sqrt{n}}, -\frac{r^n}{\sqrt{n}}\}\} = 1 \), for every \( n \). But this is a contradiction.

A p. measure \( \mu \) on \( \mathbb{R}^m \) is called sub-stable of index \( \alpha \) if there exists a symmetric \( S(\beta) \) p. measure \( v \) on \( \mathbb{R}^m \) such that \( \hat{\mu}(y) = \exp\{-\log \hat{v}(y)\}^q \), where \( \beta < \alpha \) and \( q = \beta/\alpha \). It is known that sub-stable p. measures exist and that they admit a very simple structure [4]. Motivated from this, we define sub-semistable p. measures and prove their existence; we also obtain ch. functions of other related
p. measures on $\mathbb{R}^m$. We begin by deriving a formula of the Laplace transform of a non-negative r-SS($a$) r.v.

**Lemma 2.3.** Let $0 < r < 1$, $0 < a < 1$, and let $Z$ be a real non-negative strictly r-SS($a$) r.v.. Let $F$ be the Levy measure of $\mathcal{L}_Z$; then the Laplace transform $\hat{L}_Z$ of $Z$ is given by

$$
\hat{L}_Z(s) = \mathbb{E}(e^{-sZ}) = \exp(-\int_0^\infty \frac{r^{-n}}{n} \left(1 - e^{-sr^{n/a}x}\right) F(dx)) ,
$$

for every $s \geq 0$.

**Proof:** Since $Z \geq 0$, we have, from Theorem 3.1 of [9], that $F$ is concentrated on $(0, \infty)$; and, from Theorem 3.1 of [11], the ch. function $\hat{\mathcal{L}}_Z$ of the law $\mathcal{L}_Z$ is given by

$$
\hat{\mathcal{L}}_Z(t) = \exp(-\int_0^\infty \frac{r^{-n}}{n} \left(1 - e^{itr^{n/a}x}\right) F(dx)) .
$$

Let $F_k = F/(r^{k/a}, \infty)$ and let $e(F_k) = e^{-F_k(R)} = \sum_{n=0}^\infty \frac{F^n_k}{n!}$, then clearly $\hat{e}(F_k)$, the Laplace transform of $e(F_k)$, is given by

$$
\hat{e}(F_k)(s) = \mathbb{E}(e^{-sF_k}) = \exp(-\int_0^\infty \frac{r^{-n}}{n} \left(1 - e^{-sr^{n/a}x}\right) F(dx)) .
$$

Now, from Corollary 1 of [3], we have $e(F_k) \sim \mathcal{L}_Z \ast \delta_a$, for some $a \in \mathbb{R}$; hence

$$
\hat{\mathcal{L}}_{e(F_k)}(t) = \exp(-\int_0^\infty \frac{r^{-n}}{n} \left(1 - e^{itr^{n/a}x}\right) F(dx)) = \hat{\mathcal{L}}_Z(t) e^{iat} .
$$

But $\hat{\mathcal{L}}_Z(t) = \exp(-\int_0^\infty \frac{r^{-n}}{n} \left(1 - e^{itr^{n/a}x}\right) F(dx)) = \lim_{k \to \infty} \hat{e}(F_k)(t)$, so $a = 0$. 
Thus, \( e(F_k)^\mathcal{N}_Z \), and hence \( \hat{L}_Z(s) = \lim_{k \to \infty} \hat{L}_e(F_k)(s) = \exp\{-\sum_n r^{-n} \int_{(r^n/\alpha, 1]} (1 - e^{-s r^n/\alpha}) F(dx)\} \), by (2.9).

With the analogy of the stable case [4,5], we make the following definition: Let \( 0 < r < 1 \) and \( 0 < \alpha < 2 \), then a symmetric \( r\text{-SS}(\alpha) \) p. measure \( \mu \) on \( \mathbb{R}^m \) will be called \( r\text{-sub-semistable of index } \alpha \), \( r\text{-sub-SS}(\alpha) \) for short, if there exists a symmetric \( r'\text{-SS}(\beta) \), \( 0 < \alpha < \beta \), and a p. measure \( \nu \) on \( \mathbb{R}^m \) such that

\[
\log \hat{\mu}(y) = (\log \hat{\nu}(y))^q, \tag{2.10}
\]

where \( q = \alpha/\beta \). We will show that given any \( \beta, r \) and \( \alpha \) satisfying \( 0 < \beta < 2 \), \( 0 < r < 1 \) and \( 0 < \alpha < \beta \), one can choose an \( r' \in (0, 1) \) such that for any given \( r'\text{-SS}(\beta) \) symmetric p. measure \( \nu \) on \( \mathbb{R}^m \), an \( r\text{-SS}(\alpha) \) p. measure \( \mu \) satisfying (2.10) exists. It is known that a p. measure \( \mu \) on \( \mathbb{R}^m \) is sub-stable of index \( \alpha \) (see [4]) if and only if

\[
\mu = \mathcal{L}_{YZ}^1/\beta \tag{2.11}
\]

where \( Y \) is a symmetric \( S(\beta) \) m-dimensional \( r \)-vector and \( Z \) is a non-negative \( S(q) \) r.v. which is independent of \( Y \) and \( \mathbb{L}_Z(s) = e^{-|s|^q} \). (If in (2.11) \( Y \) is Gaussian, then \( \mu \) is called sub-Gaussian). In the semistable case, we are unable to obtain such a representation for \( \mu \). In fact, if \( Z \) is a non-negative \( r\text{-SS}(q) \) and \( Y \) a symmetric \( r\text{-SS}(\beta) \) \( r \)-vector on \( \mathbb{R}^m \) independent of \( Z \), then it is not clear if \( \mathcal{L}_{YZ}^1/q \) is a semi-stable measure; even though, in certain special situations it is the case as will be shown in the following:

**Theorem 2.4(a).** Let \( 0 < \beta < 2 \), \( 0 < r < 1 \), and \( 0 < \alpha < \beta \); and let \( q = \alpha/\beta \) and \( r' = r^{1/q} \). Let \( \nu \) be a symmetric \( r'\text{-SS}(\beta) \) p. measure on \( \mathbb{R}^m \), then

\[
\phi(y) = \exp\{-\log \hat{\nu}(y)^q
\]

is the ch. function of an \( r\text{-SS}(\alpha) \) p. measure, say \( \mu \); further, \( \mu \) is a mixture of
the measures $v^u$, where $v^u$ denotes the $u$-th root of $v$, $u > 0$.

(b) Let $0 < r < 1$, $0 < \beta < 2$, and let $Y$ be a symmetric $S(\beta)$ $m$-dimensional $r$-vector. Let $0 < \alpha < \beta$ and $q = \alpha/\beta$. Let $Z$ be a non-negative $r$-$SS(q)$ r.v. independent of $Y$, then $X = YZ^{1/\beta}$ is an $r$-$SS(\alpha)$ $r$-vector, and, for $y \in \mathbb{R}^m$,

$$\hat{L}_X(y) = \exp\left(-\int \frac{r^n}{n} \left[1 - \exp\left(-\frac{r^n}{q} t\right) \int_{\mathbb{S}} \langle x, y \rangle \, G(dx) \, G(dt)\right] \, (r^{1/\beta}, 1) \right)$$

where $G$ and $\Gamma$ are, respectively, the spectral measures of $L_Z$ and $L_Y$. Further, the Lévy measure $F$ of $L_X$ is given by

$$F(A) = \int \frac{r^n}{n} \Lambda\left(r^{n/\alpha} A\right), \quad (2.12)$$

for every Borel set $A$, where

$$\Lambda = (G \times L_Y) \circ \psi^{-1}$$

and $\psi: (r^{1/q}, 1] \times \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\psi(t, x) = t^{1/\beta} x$.

(c) If in part (b) $Y$ is Gaussian with mean zero and covariance $K$ (hence $\beta = 2$), then the ch. function $L_X$ of $X$ is given by

$$\hat{L}_X(y) = \exp\left(-\int \frac{r^n}{n} \left[1 - \exp\left(-\frac{r^n}{q} t\{K_y, y\}\right) \, G(dt)\right] \, (r^{1/\beta}, 1) \right)$$

and the Lévy measure of $L_X$ is given by (2.12).

Proof of (a): From the Bernstein's theorem, we have $\exp(-t^q) = \int_0^\infty \exp\{-tu\} \, \tau(du)$, $t > 0$, where $\tau$ is a p. measure on $(0, \infty)$; define a p. measure $\mu$ on $\mathbb{R}^m$ by

$$\mu(A) = \int_0^\infty \nabla^u(A) \, \tau(du),$$

for every Borel set $A$ (using the integrability of $\nabla^u$, one can show that $\nabla^u(A)$
is measurable in $u$). It follows, using usual limit arguments, and using
$\hat{v}(y) = \exp(-u(-\log \hat{v}(y)))$ (see [3]), that

$$
\hat{\mu}(y) = \int_0^\infty \left( \int_{\mathbb{R}^m} e^{i \langle x, y \rangle} u(dy) \right) \tau(du)
= \int_0^\infty \exp\{-u(-\log \hat{v}(y))\} \tau(du)
= \exp\{-(-\log \hat{v}(y))^q\},
$$

for all $y \in \mathbb{R}^m$. This argument shows that $\phi$ is a ch. function of an i.d. p. measure; further, using the form $\hat{v}$, one easily verifies that $\phi^r(y) = \phi(r^{1/\alpha} y)$. Thus $\mu$ is an $r$-SS($\alpha$) p. measure; completing the proof of part (a).

Proof of (b): Using independence of $Y$ and $Z$, Corollary 2.1 of [8] and Lemma 2.3, we have, for every $y \in \mathbb{R}^m$,

$$
\hat{\mathcal{L}}_X(y) = \mathbb{E}[e^{i \langle YZ^{1/\beta}, y \rangle}] = \exp\{-Z \int_{\partial S} |\langle x, y \rangle|^{\beta} \Gamma(dx)\}
= \exp\{- \sum_n r^{-n} \int_{(r^{1/q}, 1]} [1 - \exp(-r^{n/q} t \int_{\partial S} |\langle x, y \rangle|^{\beta} \Gamma(dx))] G(dt)\},
$$

which satisfies $\hat{\mathcal{L}}_X(y) = \mathcal{L}_X(r^{1/\alpha} y)$; hence the proof of the first part is complete.

Now as we know from [8], $\exp\{-\int_{\partial S} |\langle x, y \rangle|^{\beta} \Gamma(dx)\} = \int_{\mathbb{R}^m} \cos(\langle x, y \rangle) \mathcal{L}_Y(dx)$,

for every $y \in \mathbb{R}^m$, we obtain, from (2.13),

$$
\hat{\mathcal{L}}_X(y) = \exp\{-\sum_n r^{-n} \int_{(r^{1/q}, 1]} [\int_{\mathbb{R}^m} (1 - \cos(t^{1/\beta} x, r^{n/\alpha} y)) \mathcal{L}_Y(dx)] G(dt)\}
= \exp\{-\sum_n r^{-n} \int_{\mathbb{R}^m} (1 - \cos(\langle x, r^{n/\alpha} y \rangle) \Lambda(dx)\} ;
$$

hence, in view of the converse part of Theorem 3.1 of [11], the proof of part (b) will be complete if we can show that $\int_{\mathbb{R}^m} \|x\|^\alpha \Lambda(dx) < \infty$, but this is true, since
\[
\int_{\mathbb{R}^m} \|x\|^\alpha A(dx) = \left( \int_{(r/\gamma, 1]} \|x\|^\alpha \mathbb{L}_Y(dx) \right) < \infty
\]
(recall $\alpha < \beta$, and $Y$ is $S(\beta)$). Completing the proof of (b), the proof of (c)
is similar.

REMARK 2.5. We have already noted that, unlike the stable case, we are unable
to find any structural representation for sub-semistable $p.$ measures. Another
interesting question which we have not addressed here is the characterization of the
so called spherically generated [5] semistable $p.$ measures; in the stable case, as
is well known [5], this class coincides with the class of sub-Gaussian $p.$ measures.
With this analogy one may wonder if the class of $p.$ measures $\mathbb{L}_X$ of Theorem 2.4(b)
(each $\mathbb{L}_X$ can be easily shown to be spherically generated) exhaust the class of
all spherically generated semistable $p.$ measures. This is false, since the Lévy
measure $F$ of $\mathbb{L}_X$ is easily shown to satisfy $F \ll \text{Leb.}$, and one can construct
rotationally invariant (and hence spherically generated) $p.$ measure $\mu$ whose Lévy
measure is not absolutely continuous with respect to the Lebesgue measure.

3. LINEAR REGRESSION FOR I.D. AND SEMISTABLE $P.$ MEASURES

Let $X = (X_0, X_1, \ldots, X_m)$ be an $m + 1$ dimensional i.d. random vector with
$E(X_j) = 0$, for $j = 0, \ldots, m$; we will provide necessary and sufficient condition,
in terms of the Lévy measure of $\mathbb{L}_X$, in order that $E(X_0/X_1, \ldots, X_m)$ is a linear
function of $X_1, \ldots, X_m$ (this property is referred to as the linear regression).
We will also provide a counterexample showing that linear regression may fail for a
systematic r-SS($\alpha$), $1 < \alpha < 2$, r. vector $X$ even when $m = 1$; this is in con-
trast to the symmetric stable 2-dimensional r. vector in which case linear regression
always holds. The results in this section are motivated from the work of Miller
[10] and Kantor [6], who studied the linear regression property for the symmetric
stable r. vectors.

The following result is the generalization of Corollary 3.2 of [10] to i.d.
random vectors $X$. In our proof we make use of a known general condition, in terms
of ch. function of $X$, for linear regression; this condition was first used by
Miller [10] while studying linear regression for symmetric stable r. vectors.
THEOREM 3.1. Let \( X = (X_0, X_1, \ldots, X_m) \) be an i.d. r. vector with \( \|x\| \mathcal{L}_X(dx) < \infty \) and \( E(X) = 0 \). Then

\[
E(X_0/X_1, \ldots, X_m) = \sum_{j=1}^{m} a_j X_j,
\]

(3.1)

for some \( a_1, \ldots, a_m \), if and only if

\[
\int_{RXA} (x_0 - \sum_{j=1}^{m} a_j x_j)^+ F(dx) = \int_{RXA} (x_0 - \sum_{j=1}^{m} a_j x_j)^- F(dx),
\]

(3.2)

for every \( A \in \mathcal{B}(R^m \setminus \{0\}) \), where \( F \) is the Levy measure of \( \mathcal{L}_X \).

For the proof of the above theorem we will need the following essentially known Lemma.

LEMMA 3.2. Let \( \mu \) be a generalized Poisson measure on \( R^m \) with Levy measure \( F \). If \( \int_{R^m} \|x\| \mu(dx) < \infty \) and \( E(X) = 0 \), then, for every \( y \in R^m \),

\[
\hat{\mu}(y) = \exp\left\{ \int_{R^m} (e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle) F(dx) \right\} ;
\]

further, \( \partial \hat{\mu}(y) / \partial y_j \) exists, and

\[
\frac{\partial \hat{\mu}(y)}{\partial y_j} = \hat{\mu}(y) \int_{R^m} i x_j (e^{i\langle x, y \rangle} - 1) F(dx),
\]

for every \( j = 1, 2, \ldots, m \).

Proof: It is known (see, e.g., Kruglov [7, p. 408]) that \( \int_{R^m} \|x\| \mu(dx) < \infty \) \( \Rightarrow \int_{\{\|x\| \geq 1\}} \|x\| F(dx) < \infty \). Hence, since \( \int_{\{\|x\| \geq 1\}} |\langle x, y \rangle| F(dx) \leq \int_{\{\|x\| \geq 1\}} \|x\| F(dx) \), \( a \equiv \int_{\{\|x\| \leq 1\}} x F(dx) \) is defined as an element of \( R^m \).

It follows, from Tortrat [12, p. 88], that for some \( b \in R^m \),

\[
\hat{\mu}(y) = \exp\left\{ \int_{R^m} (e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle I_{\{x \leq 1\}}(x) F(dx) \right\} e^{i\langle b, y \rangle}
\]

\( \{\|x\| \leq 1\} \)
for every \( y \in \mathbb{R}^m \). Now the condition \( \int_{\mathbb{R}^m} \|x\| \mu(dx) < \infty \) implies that \( \frac{\partial \hat{\mu}}{\partial y_j} \)
exists and that \( \int_{\mathbb{R}^m} x_i \mu(dx) = a + b = 0 \) (see, e.g., [1, p. 265]). Thus \( \hat{\mu} \) has the desired representation and

\[
\frac{\partial \hat{\mu}(y)}{\partial y_j} = \hat{\mu}(y) \frac{a}{\partial y_j} \left[ \int_{\mathbb{R}^m} \{ e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle \} F(dx) \right].
\]

To complete the proof the only thing needed to be verified is that the operations of integration and differentiation can be interchanged in (3.3). But this follows from the following inequalities:

\[
\sup_{\|y\| \leq C} \left| \frac{\partial}{\partial y_j} (e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle) \right| \leq \|x\| \min(2, C\|x\|),
\]

for any \( C > 0 \); and

\[
\int_{\mathbb{R}^m} \|x\| \min(2, C\|x\|) F(dx) < \infty.
\]

Proof of Theorem 3.1: According to Theorem 3.1 of [10], (3.1) holds \( \Leftrightarrow \) for all \( y_1, \ldots, y_m \),

\[
\frac{a}{\partial y_0} \phi(y_0, y_1, \ldots, y_m) = \sum_{j=1}^{m} a_j \frac{a}{\partial y_j} \phi(0, y_1, \ldots, y_j, \ldots, y_m),
\]

where \( \phi = \mathcal{L}_X \). This, by Lemma 3.1, is equivalent to

\[
\int_{\mathbb{R}^{m+1}} x_0 \left( e^{i \sum_{j=1}^{m} x_j y_j} - 1 \right) F(dx) = \int_{\mathbb{R}^{m+1}} \sum_{j=1}^{m} a_j x_j \left( e^{i \sum_{j=1}^{m} x_j y_j} - 1 \right) F(dx),
\]

for every \( y_1, \ldots, y_m \); or
\[ \int_{\mathbb{R}^{m+1}} (x_0 - \sum_{j=1}^{m} a_j x_j)(e^{i \sum_{j=1}^{m} x_j y_j} - 1) \, F(dx) = 0, \quad (3.4) \]

for every \( y_1, \ldots, y_m \). For \( x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} \), define

\[ f(x) = x_0 - \sum_{j=1}^{m} a_j x_j. \]

Let \( \xi = (y_1, \ldots, y_m) \) and \( \eta = (z_1, \ldots, z_m) \) be two arbitrary vectors in \( \mathbb{R}^m \); then, by (3.4), we have

\[ \int_{\mathbb{R}^{m+1}} \left( (e^{i \langle x, \xi + \eta \rangle} - 1) + (e^{i \langle x, \xi - \eta \rangle} - 1) - 2(e^{i \langle x, \xi \rangle} - 1) f(x) \right) F(dx) = 0, \]

where \( \pi \) is the natural projection. This yields

\[ \int_{\mathbb{R}^{m+1}} e^{i \langle x, \xi \rangle} (1 - \cos \langle x, \eta \rangle) \, f(x) \, F(dx) = 0; \]

thus

\[ \int_{\mathbb{R}^{m+1}} e^{i \langle x, \xi \rangle} (1 - \cos \langle x, \eta \rangle) \, f^+(x) \, F(dx) = \]

\[ = \int_{\mathbb{R}^{m+1}} e^{i \langle x, \xi \rangle} (1 - \cos \langle x, \eta \rangle) \, f^-(x) \, F(dx); \]

which implies

\[ \int_{\mathbb{R}^m} e^{i \langle x, \xi \rangle} \nu^+_\eta \circ \pi^{-1}(dx) = \int_{\mathbb{R}^m} e^{i \langle x, \xi \rangle} \nu^-_\eta \circ \pi^{-1}(dx), \quad (3.5) \]

where \( \nu^+_\eta(dx) = (1 - \cos \langle x, \eta \rangle) f^+(x) \, F(dx) \) and

\( \nu^-_\eta(dx) = (1 - \cos \langle x, \eta \rangle) f^-(x) \, F(dx) \). Since

\[ \int_{\mathbb{R}^{m+1}} |(1 - \cos \langle x, \eta \rangle) f(x)| F(dx) \]

\[ \leq C_1 \int_{\{\|x\| \leq 1\}} \|x\|^2 F(dx) + C_2 \int_{\{\|x\| > 1\}} \|x\| F(dx) < \infty, \]
(where \( C_1 \) depends only on \( \eta \) and \( a_j \)'s, and \( C_2 \) depends only on \( a_j \)'s), both \( \nu^+_{\eta} \) and \( \nu^-_{\eta} \) are finite measures. Hence, by the uniqueness of Fourier transform, we have, from (3.5),

\[
\int_{\mathbb{R}^n} (1 - \cos(\pi x, \eta)) f^+(x) \, F(dx) = \int_{\mathbb{R}^n} (1 - \cos(\pi x, \eta)) f^-(x) F(dx) \quad (3.6)
\]

for every \( B \in B(\mathbb{R}^n) \). As \( \eta \) was arbitrary, using a standard argument, (3.6) finally yields (3.2).

REMARK 3.3. In general one cannot replace (3.1) by \( \int_{\mathbb{R}^n} f(x) \, F(dx) = 0 \), as \( \int_{\mathbb{R}^n} f(x) \, F(dx) \) may not exist. A simple example which makes this point is the following: Let

\[
F = \sum_{n=1}^{\infty} \{2^n \delta_{(0,2^{-n})} + 2^n \delta_{(2^{-n}, 2^{-n})}\},
\]

then \( F \) is Levy on \( \mathbb{R}^2 \); and for any \( A = \{2^n \} \), we have

\[
\int_{\mathbb{R}^2} (x_0 - \frac{1}{2} x_1)^+ \, dF = \int_{\mathbb{R}^2} (x_0 - \frac{1}{2} x_1)^- \, dF = \frac{1}{2}.
\]

However, if \( B = \{2^n: n = 1,2,\ldots\} \), then

\[
\int_{\mathbb{R}^2} (x_0 - \frac{1}{2} x_1)^+ \, dF = \int_{\mathbb{R}^2} (x_0 - \frac{1}{2} x_1)^- \, dF = \infty.
\]

Using Theorem 3.2 and a criterion of independence for i.d. random variables, (Theorem 6.1 of [11]), one easily obtains the following corollary which for the special case for symmetric stable r.v.'s was obtained by Kantor [6] (see also Miller [10]). Since the proof is rather straightforward, it will be omitted.

COROLLARY 3.4. Let \( X = (X_0, X_1, \ldots, X_m) \) satisfy the hypotheses of Theorem 3.1; assume, in addition, that \( E(X_0/X_j) = a_j X_j \), for some \( a_j, j = 1, \ldots, m \) and that \( X_1, \ldots, X_m \) are pairwise independent; then
Now we provide two examples: The first shows that linear regression in general does not hold for two non-symmetric r.v.'s; this is in contrast to the case of the two symmetric stable r.v.'s when linear regression always holds; the second example shows that linear regression may fail for two r-SS(α) symmetric r.v.'s.

EXAMPLE 3.5(i). Let $X_0, X_1$ be strictly stable r.v.'s of index $\alpha$ with $1 < \alpha < 2$; and let $\Gamma$ be the spectral measure on the sphere $\partial S$ of $\mathbb{R}^2$ of the $\kappa_X$ of $X = (X_0, X_1)$. Then

$$E(X_0/X_1, \ldots, X_m) = \sum_{j=1}^{m} a_j X_j.$$  

Now let $\Gamma = \delta(0,1) + \delta(0,-1) + \delta\left( \frac{1}{\sqrt{1+c^2}}, \frac{-c}{\sqrt{1+c^2}} \right) + \delta\left( \frac{-1}{\sqrt{1+c^2}}, \frac{-c}{\sqrt{1+c^2}} \right)$.

(iii) Choose $c$ such that $c \neq \left( r^{-2n/\alpha} - 1 \right)^{1/2}$, for any $n = 1, 2, \ldots$. Let

$$\Gamma = \delta(0,1) + \delta(0,-1) + \delta\left( \frac{1}{\sqrt{1+c^2}}, \frac{-c}{\sqrt{1+c^2}} \right) + \delta\left( \frac{-1}{\sqrt{1+c^2}}, \frac{-c}{\sqrt{1+c^2}} \right).$$

it follows, using Theorem 3.1 of [10], that $E(X_0/X_1) = aX_1$, for some $a > 0$.

$$\int_{\partial S} \text{sgn}(x_1) |x_1|^{\alpha-1} x_0 \, d\Gamma = a \int_{\partial S} \text{sgn}(x_1) |x_1|^{\alpha-1} x_1 \, d\Gamma$$  (3.7)

and

$$\int_{\partial S} |x_1|^{\alpha-1} x_0 \, d\Gamma = a \int_{\partial S} |x_1|^{\alpha-1} x_1 \, d\Gamma.$$  (3.8)

Now let $\Gamma = \delta(2^{-1/2}, 2^{-1/2}) + \delta(2^{-1/2}, -2^{-1/2})$ and the right side is $0$ for any $a$. Thus (3.8) fails; (similarly, (3.7) fails).

(ii) Let $0 < r < 1, 1 < \alpha < 2$; choose $c$ such that $c \neq \left( r^{-2n/\alpha} - 1 \right)^{1/2}$, for any $n = 1, 2, \ldots$. Let

$$\Gamma = \delta(0,1) + \delta(0,-1) + \delta\left( \frac{1}{\sqrt{1+c^2}}, \frac{-c}{\sqrt{1+c^2}} \right) + \delta\left( \frac{-1}{\sqrt{1+c^2}}, \frac{-c}{\sqrt{1+c^2}} \right).$$
and let \( X = (X_0, X_1) \) be (symmetric) \( r \)-SS\((\alpha) \) \( r \)-vector with spectral measure \( \Gamma \); and let \( F \) be the corresponding Lévy measure (see Theorem 1.1). We assert that

\[
E(X_0/X_1) \neq aX_1, \tag{3.9}
\]

for every \( a \). To see this we make use of (3.2). First let \( a > 0 \), then

\[
\int_{RX\{1\}} (x_0 - ax_1)^+ dF = \int_{RX\{1\}} (x_0 - a)^+ dF = 0, \text{ since } F(\{x_0, 1\} : x_0 > a) = F(\{x_0, cx_0\} : x_0 = \frac{r^{-n/a}}{\sqrt{1+c^2}}, \text{ for some } n, \text{ and } cx_0 = 1) = 0 \text{ by the choice of } c. \]

On the other hand,

\[
\int_{RX\{1\}} (x_0 - ax_1)^- dF = \int_{RX\{1\}} (-a)^- dF = aF(\{0, 1\}) = a > 0.
\]

Thus (3.9) holds for any \( a > 0 \); similar calculations show that (3.9) holds for any \( a < 0 \); finally if \( a = 0 \), then

\[
\int_{RX\{\frac{-c}{\sqrt{1+c^2}}\}} x_0^+ dF = \frac{1}{\sqrt{1+c^2}} \quad \text{and} \quad \int_{RX\{\frac{c}{\sqrt{1+c^2}}\}} x_0^- dF = 0. \]

Thus linear regression does not hold.

REFERENCES


Hermite Expansions of Generalized Brownian Functionals
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1. Introduction. In the paper [6], Kuo has used Hida's theory of generalized Brownian functionals to give a rigorous meaning to the so called Donsker's delta function. It is defined formally as

\[ \delta_{t,x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iy(\xi - x)} \, dy, \]

where \( B_t \) is a Brownian motion. Equivalently \( \delta_{t,x} \) is a composition of Dirac's delta distribution with \( B_t \). Obviously such a composition can not be realized as an ordinary Brownian functional i.e. an element of \( L^2_{\mathcal{G}}(\sigma(B_t, t \in \mathbb{R}^3)) \).

The analogous problem of defining a composition of Schwartz's distribution with a Brownian functional was discussed in [5,9] and recently in [4].

In this paper we give a natural construction of \( u(B_t) \), for \( u \in \mathcal{S}' \), based on the expansion of the distribution \( u \) in Hermite functions. If \( u \in \mathcal{S}' \) then \( u \) has the \( \mathcal{S}' \)-convergent expansion

\[ u = \sum_{n=0}^{\infty} a_n h_n, \]

where \( h_n \) is the \( n \)-th Hermite polynomial given by (3.1) below. The composition \( u(B_t) \) can then be defined by

\[ u(B_t) = \sum_{n=0}^{\infty} a_n h_n \left( \frac{B_t}{t} \right). \]

We prove that this series is convergent in the Hida space \( (L^2)^- \) so that \( u(B_t) \) makes sense as a generalized Brownian functional. As an application we derive some formulas which appeared in [8], and [5].

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2. Generalized Brownian functionals.

We give here a brief description of Hida's theory of generalized Brownian functionals [2]. For a more complete account see e.g. [2,7].

Our basic probability space is \((\mathcal{S}', \mu)\) where \(\mathcal{S}' = \mathcal{S}'(\mathbb{R})\) is the space of tempered distributions and \(\mu\) is the standard white noise measure on \(\mathcal{S}'\) i.e.

\[
\int_{\mathcal{S}'} e^{i\langle x, \xi \rangle} \mu(dx) = \exp\left(-\frac{1}{2}||\xi||^2\right), \quad \xi \in \mathcal{S}'
\]

where \(||\cdot||\) denotes the \(L^2(\mathbb{R})\) norm.

The \(S\)-transform of a Brownian functional \(\phi \in L^2(\mathcal{S}') = L^2(\mathcal{S}', \mu)\) is defined by the formula

\[
S\phi(\xi) = \int_{\mathcal{S}'} \phi(x + \xi) \mu(dx)
= \exp\left(-\frac{1}{2}||\xi||^2\right) \int_{\mathcal{S}'} e^{i\langle x, \xi \rangle} \phi(x) \mu(dx).
\]

The space \(L^2(\mathcal{S}')\) has the well-known Wiener-Ito decomposition

\[
L^2(\mathcal{S}') = \bigoplus_{n=0}^{\infty} K_n
\]

where \(K_n\) is the \(n\)-th homogeneous chaos. If \(\phi \in K_n\) then we have

\[
S\phi(\xi) = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \xi(t_1) \cdots \xi(t_n) dt_1 \cdots dt_n
\]

where \(f \in L^2(\mathbb{R}^n)\) (symmetric \(L^2(\mathbb{R}^n)\) functions). The map \(\phi \mapsto f\) from \(K_n\) into \(L^2(\mathbb{R}^n)\) is surjective; for \(f \in L^2(\mathbb{R}^n)\) the corresponding \(\phi\) is the multiple Wiener integral

\[
I_n(f) = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) dB_{t_1} \cdots dB_{t_n}.
\]

If \(\phi \in L^2(\mathcal{S}')\) then \(\phi = \sum_{n=0}^{\infty} \phi_n\), \(\phi_n = I_n(f_n)\) for some \(f_n \in L^2(\mathbb{R}^n)\) and the map \(\phi \mapsto (f_n)_{n=0}^{\infty}\) is unitary onto the weighted space \(\bigoplus_{n=0}^{\infty} \sqrt{n!} L^2(\mathbb{R}^n)\) i.e.

\[
\sum_{n=0}^{\infty} \sqrt{n!} f_n^2 \leq L^2(\mathcal{S}')
\]
Let \((H^\alpha(\mathbb{R}^d), \| \cdot \|_\alpha)\), \(\alpha\) real, denote the Sobolev space of order \(\alpha\),
\[ \| f \|_\alpha^2 = \int_{\mathbb{R}^d} |f(\gamma)|^2 (1 + |\gamma|^2)^{\alpha} d\gamma \]
where \(f\) denotes the Fourier transform of \(f\) and
\[ H^\alpha(\mathbb{R}^d) = \{ f \in \mathcal{S}' ; \| f \|_\alpha < +\infty \}. \]

The spaces \(H^\alpha(\mathbb{R}^d)\) and \(H^{-\alpha}(\mathbb{R}^d)\) are dual to one another in the sense of Banach spaces.

The space of test functionals \(L^2_+ \subset L^2(\mathcal{S}')\) is defined as follows:
\[ \phi \in (L^2)_+ \text{ iff the corresponding sequence } (f_n)_{n=0}^\infty, \phi = \sum_{n=0}^\infty f_n, \text{ is such that } f_n \in H^{\frac{n+1}{2}}(\mathbb{R}^d) \text{ and } \| \phi \|_+^2 = \sum_{n=0}^\infty n! \| f_n \|_{\frac{n+1}{2}}^2 < +\infty. \]

The dual space to \((L^2)_+\) is the space of generalized Brownian functionals and is denoted by \((L^2)_-\). We have the natural embeddings
\[ (L^2)_+ \subset L^2(\mathcal{S}') \subset (L^2)_-. \]
To each \(\phi \in (L^2)_-\) there corresponds a sequence \((f_n)_{n=0}^\infty, f_n \in H^{\frac{n+1}{2}}\) and we have
\[ \| \phi \|_-^2 = \sum_{n=0}^\infty n! \| f_n \|_-^{n+1}. \]

If \(\psi \in (L^2)_+\) is represented by \((g_n)_{n=0}^\infty, g_n \in H^{\frac{n+1}{2}}(\mathbb{R}^d) = H^{\frac{n+1}{2}} \wedge (\mathbb{R}^d)\)
and \(\phi \in (L^2)_-\) is represented by \((f_n)_{n=0}^\infty, f_n \in H^{\frac{n+1}{2}}(\mathbb{R}^d)\) then
\[ \langle \phi, \psi \rangle = \sum_{n=0}^\infty \langle f_n, g_n \rangle. \]
Here \(\langle \cdot, \cdot \rangle\) denotes the canonical pairing between \((L^2)_-\) and \((L^2)_+\).
\[ \langle \phi, \psi \rangle = \mathbb{E}\psi \text{ if } \phi, \psi \in L^2(\mathcal{S}'). \]
3. Compositions with tempered distributions.

For $f \in L^2(\mathbb{R})$, $I(f) = I_1(f)$ i.e. $I(f)$ is the Wiener integral of $f$. We will define the composition $u(I(f))$ for arbitrary $u$ in $\mathscr{S}'$.

Let $h_n$, $n \geq 0$ denote the $n$-th Hermite polynomial, $h_0 = 1$, and

$$h_n(x) = \frac{(-1)^n e^{x^2/2}}{(n!)^{1/2}} n! e^{-x^2/2}.$$  

If $u \in \mathscr{S}'$ and $\sigma > 0$ then $u$ has an $\mathscr{S}'$ convergent expansion

$$u = \sum_{n=0}^{\infty} a_n h_n(x),$$

with $a_n = \langle u, g_n, \sigma \rangle$, $g_n, \sigma = h_n(X) * g_\sigma$, where $g_\sigma(x) = (\sqrt{2\pi} \sigma)^{-1} \exp(-x^2/2\sigma^2)$.

This expression is obtained by expanding the distribution $u * (g_\sigma)^{1/2}$ in terms of Hermite functions

$$H_n(X) \sigma = h_n(X) (g_\sigma)^{1/2}.$$  

It is well known that the $H_n$'s form a basis in $\mathscr{S}'$ (see e.g. [e.g. [10], p. 267 or [1]), moreover there exists $N > 0$, depending on $u$ such that

$$\left| a_n \right| < \text{const} \cdot (1 + n)^N.$$  

The composition $u(I(f))$ is defined by

$$u(I(f)) = \sum_{n=0}^{\infty} a_n h_n(I(f) \sigma)$$

with $\sigma = \|f\|$. The series is $(L^2)_{-}$ convergent according to Theorem 1 below. In particular for $f = \chi_{[0,t]}$ and $u = \delta_x$ we have the expansion of $\delta_{t,x} = \delta_x(B_t)$ discussed by Kuo [6].

$$\delta_{t,x} = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \sum_{n=0}^{\infty} h_n \left( \frac{x}{\sqrt{t}} \right) h_n \left( \frac{B_t}{\sqrt{t}} \right).$$

**Theorem 1.** The series in (3.4) is convergent in the space $(L^2)_{-}$. Moreover we have

$$su(I(f))(\xi) = (u * g_\sigma)(<f,\xi>),$$

where $g_\sigma(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp(-x^2/2\sigma^2)$. 


Proof. The random variable $h_n(f)$ is in $K_n$ and is represented by (see e.g. Hida [2], p. 139)

\begin{equation}
 f_n = \left(\sqrt{n!} \sigma^2\right)^{-1} \bigotimes_n n^\alpha
\end{equation}

Therefore we need to show that

\[ \sum_{n=0}^{\infty} n^2 \int_{|f|} 2n \bigotimes_n n^2 - \frac{n+1}{2} < +\infty \]

but this follows from (3.3) and the following elementary lemma,

Lemma 2. For every $f \in L^2(\mathbb{R}^n)$ there exists $0 < q < 1$ such that

\[ \|f \|_{n} < \text{const} q^n \|f\|^n \]

Proof. Let $R > 0$. We have from the definition of $\bigotimes_n n^\alpha$,

\[ \|f \|_{n} < \int_{|y|} \left| f(y) \right|^2 dy + \frac{1}{(1 + R^2)^\alpha} \cdot \bigotimes_n n^\alpha \]

Obviously there exists $R_0 > 0$ such that for $R < R_0$, the first integral is no greater than $2^{-n} \bigotimes_n 2n = 2^{-n} \|f\| 2n$. Now

\[ \|f \|_{n} < (2^{-n} + q^n) \|f\| 2n \]

with $q_1 = \frac{1}{1 + R_0^2}$. For $\alpha = \frac{n+1}{2}$ the lemma follows with $q = \sqrt{q_1}$.

To prove (3.6) we compute the $S$ transform of $\phi = u(I(\gamma))$ as follows: according to (3.7)

\[ S\phi(\xi) = \sum_{n=0}^{\infty} a_n \frac{1}{\sqrt{n!} \sigma} \left( \bigotimes_n n, \xi \bigotimes_n n \right) L_2(\mathbb{R}) \]

\[ = \sum_{n=0}^{\infty} u \cdot g_n, \sigma > \frac{1}{\sqrt{n!} \sigma} (f, \xi)^n. \]

Now using the Hermite generating function

\[ \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{n!}} h_n(x) = e^{\gamma x - \gamma^2 / 2} \]
and setting $\beta = (f, \xi)$, we obtain

$$S_4(\xi) = \langle f, \frac{1}{\sqrt{2\pi \sigma}} \exp - \frac{1}{2\sigma^2} (x - \beta)^2 \rangle.$$ 

which proves (3.7).

Remark. In Kubo [5], the right hand side of (3.6) was used to define $u(I(f))$.

We now discuss the dependence on parameter in $u(I(f))$.

Let $\mathcal{H}_N$ denote the Hilbert space

$$\mathcal{H}_N = \{ u \in \mathcal{H}; \sum_{n=0}^{\infty} |\langle u, H_n \rangle|^2 (n + 1)^{2N} < \infty \}.$$ 

Where $H_n$ is the $n$-th Hermite function given as in (3.2). We have $\mathcal{H}_N = \mathcal{H}_{-N}$ and it is well known that $\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N$ and $\mathcal{H}' = \bigoplus_{N=0}^{\infty} \mathcal{H}_{-N}'$. The sequence $v_k$ converges strongly in $\mathcal{H}'$ to $v_0$ if there exists $N > 0$ such that

$$\{v_k, k > 0\} \subset \mathcal{H}_{-N} \text{ and the convergence takes place in } \mathcal{H}_{-N}.$$ 

Let $u_k = \sum_{n=0}^{\infty} a_n^{(k)}(\gamma_n)$, $k = 0, 1, \ldots$ be the expansion of $u_k$ in $(h_n)$. Applying above $u_k(g_o)^{1/2}$, $k = 0, 1, \ldots$ we obtain that $u_k + u_0$ strongly in $\mathcal{H}'$ iff there exists $N > 0$ such that

$$(3.8) \quad \lim_{k \to \infty} \sum_{n=0}^{\infty} |a_n^{(k)} - a_n^{(0)}|^2 (n + 1)^{-N} = 0.$$ 

Proposition 3. If $u_k + u_0$ strongly in $\mathcal{H}'$ as $k \to \infty$ then $u_k(I(f)) + u_0(I(f))$ in $(L^2)^\prime$. 

Proof. In the same way as in the proof of the first part of Theorem 1 we get

$$\|u_k(I(f)) - u_0(I(f))\|_{L^2}^2 =$$

$$\sum_{n=0}^{\infty} |a_n^{(k)} - a_n^{(0)}|^2 \left[ \left( \sum_{n=0}^{n} a_n^{(k)} \right)^2 - \frac{n+1}{2} \right] \leq \text{const} \cdot \sum_{n=0}^{\infty} |a_n^{(k)} - a_n^{(0)}|^2 \cdot q^n.$$
where $0 < q < 1$ is the constant from Lemma 2. Therefore

$$
I_{1,n}(I(f)) - u_0(I(f))I_n^2 < \sum_{n=0}^{\infty} \left| a_n(0) - a_n \right|^2 (1 + n)^{-N}
$$

and the proposition follows from (3.8).

**Corollary.** Let $g_\varepsilon$ be a regularizing sequence e.g.

$$
g_\varepsilon(x) = (2\pi\varepsilon)^{-1/2}\exp(-x^2/2\varepsilon) \quad \text{and for } u \in \mathcal{S}, \quad \text{set } u_\varepsilon = u \ast g_\varepsilon.
$$

For all $u \in \mathcal{S}'$,

$$
\lim_{\varepsilon \to 0} u_{\varepsilon}(B_t) = u(B_t)
$$

in the sense of $(L^2)_-$ convergence.

In the same way as in Proposition 3 we can prove

**Proposition 4.** If $u_n$ converges strongly in $\mathcal{D}'$ to $u \in \mathcal{S}'$ then

$$
\lim_{n \to \infty} u_n(B_t) = u(B_t)
$$

in $(L^2)_-$ uniformly in $t$ on any interval $[0, T]$.

**4. Applications.**

The $\mathcal{S}'$ valued function $\delta_x$ is obviously continuous in the $x$ variable. Therefore for any $\phi \in (L^2)^*$ the function

$$
(4.1) \quad g(x) = \frac{\langle \delta_x B_t, \phi \rangle}{\langle \delta_x, 1 \rangle}
$$

is also continuous on $\mathbb{R}$.

It has been proved in [7] that $g(x)$ is a version of $E(\phi|B_t = x)$. Below we give a simple proof of this fact.

**Proposition 5.** Let $t > 0$. For any $B_t$ measurable functional $\psi$ in $L^2(\mathcal{S}^\prime)$ there holds

$$
E\psi(\phi) = E\phi(B_t).
$$

**Proof.** Let
\( (4.2) \)
\[
\psi = \sum_{n=0}^{\infty} b_n \frac{B_t}{\sqrt{t}} \prod_{n} x_{[0,t]}^{n}
\]
so that \( \psi \) is represented by
\[
\left( b_n \left( \frac{\sqrt{t}}{\sqrt{n!}} \right)^n \prod_{n} x_{[0,t]}^{n} \right)_{n=0}^{\infty}
\]

Suppose that \( \phi \) is represented by \( \left( \frac{1}{\sqrt{n!}} g_n \right)_{n=0}^{\infty} \), \( g_n \in L^2(\mathbb{R}^n) \). We have
\[
(4.3) \quad E\phi \psi = \sum_{n=0}^{\infty} a_n \left( \frac{\sqrt{t}}{\sqrt{n!}} \right)^n (g_n \prod_{n} x_{[0,t]}^{n}) L^2(\mathbb{R}^n).
\]

On the other hand from (3.4), \( \delta (B_t) \) is represented by
\[
\left( \left( \frac{\sqrt{t}}{\sqrt{n!}} \right)^{-\infty} g_n \prod_{n} x_{[0,t]}^{n} \right)_{n=0}^{\infty}
\]

Using this we obtain
\[
g(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} c_n \prod_{n} x_{[0,t]}^{n}
\]
with \( c_n = \left( g_n \prod_{n} x_{[0,t]}^{n} \right)_{L^2(\mathbb{R}^n)} \). It follows that \( g(B_t) \) is represented by
\[
\left( c_n \left( \frac{\sqrt{t}}{\sqrt{n!}} \right)^{-n} \frac{1}{\sqrt{n!}} \prod_{n} x_{[0,t]}^{n} \right)_{n=0}^{\infty}
\]

Now
\[
Eg(B_t)\psi = \sum_{n=0}^{\infty} b_n \frac{B_t}{(\sqrt{t})^n} \frac{c_n}{(\sqrt{t})^n} \left( \prod_{n} x_{[0,t]}^{n} \right) \left( \prod_{n} x_{[0,t]}^{n} \right)
\]
which together with (4.3) proves the proposition.

**Proposition 6.** ([4]). For \( \phi \in (L^2)^+ \) define
\[
g_{t,\phi}(x) = \left< \delta (B_t), \phi \right>.
\]

Then \( g_{t,\phi} \in \Psi \).

**Proof.** From (3.5) and [2] p. 139, \( \delta (B_t) \) is represented by \( (f_{t,n})_{n=0}^{\infty} \) where
Suppose that \( \phi \) is represented by \( \left( \frac{1}{\sqrt{n!}} g_n \right)_n \) so that

\[
(4.4) \quad \sum_{n=0}^{\infty} \frac{1}{2} c_n^2 < +\infty
\]

From (2.2) we get

\[
g_{t, \phi}(x) = g_{t, n}(x) \sum_{n=0}^{\infty} c_n \sqrt{t}^{-n} n! \left( \frac{x}{\sqrt{t}} \right)
\]

with

\[
c_n = \left( \sum_{n=0}^{\infty} g_n \right), \quad g_n.
\]

The function \( g_{t, n} \) is in \( J \) and the series in the formula for \( g_{t, \phi} \) is the Fourier-Hermite series. To prove that it is in \( J \) it is enough to show that the coefficients \( c_n \) are rapidly decreasing i.e., for every \( N > 0 \) there holds

\[
(4.5) \quad \left| \left( \sqrt{t} \right)^{-n} c_n \right| < \text{const} \cdot (1 + n)^{-N}
\]

(see e.g. [11] p. 100).

We have ([12] p. 100) with \( \alpha_n = \frac{n+1}{2} \)

\[
|c_n| < \int_{\mathbb{R}^n} |g_n| |y| \alpha_n \beta_n \alpha_n
\]

From Lemma 2 and (4.4)

\[
|c_n| < (\sqrt{t})^n \cdot \text{const} \cdot q^n
\]

which obviously proves (4.5) and therefore the proposition.

5. \textit{Ito's Lemma}.

As another application we will give a simple proof of Ito's Lemma for generalized processes of the form \( \phi_t = f(B_t) \) where \( f \in \mathcal{L}^j \). It has been obtained by Kubo [5].

The Ito integral \( \int \phi_t dB_t \) for \( \phi_t \) in \( (L^2) \) was defined by Kubo in [5] by the formula
\[ \int_0^T \hat{\phi}_t \, dB_t := \int_0^T \hat{\phi}_t^* \, dt \]

where \( \hat{\phi}_t^* \) is the creation operator (see [5] also [3,7]). For nonanticipating \( \hat{\phi}_t \) in \( L^2(\mathcal{M}) \) this integral coincides with the ordinary Ito integral.

It is not hard to prove that

\[ \int_0^T \hat{\phi}_t^* \, dt = \int_0^T \hat{\phi}_t \, dt \]

Theorem 7. For \( f \in \mathcal{M} \) we have

\[ f(B_t) = \int_0^T \hat{\phi}_t^* f(B_t) \, dt + \frac{1}{2} \int_0^T f''(B_t) \, dt \]

The proof is obtained by taking the regularization \( f_\varepsilon = g_\varepsilon \ast f \), applying classical Ito's lemma to \( f_\varepsilon \) and taking limits using Proposition 4 and (5.1).

References


SOME CENTRAL LIMIT THEOREMS FOR RANDOMLY INDEXED SEQUENCES
OF RANDOM VECTORS

by

Eugeniusz RYCHLIK

1. INTRODUCTION.

Let \( \{Z_n = (Z_{1,n}, Z_{2,n}, \ldots, Z_{k,n}), n \geq 1\} \) be a sequence of random vectors defined on a probability space \((\Omega, \mathcal{F}, P)\) with values in \(\mathbb{R}^k\) equipped with its Borel \(\sigma\)-field \(\mathcal{B}\).

Let \( \{N_n = (N_{1,n}, N_{2,n}, \ldots, N_{k,n}), n \geq 1\} \) be a sequence of random vectors defined on the same probability space \((\Omega, \mathcal{F}, P)\) with values in \(\mathbb{N}^k\), where \(\mathbb{N}\) is the set of natural numbers.

Define

\[
Z_{N_n} = (Z_{1,N_{1,n}}, Z_{2,N_{2,n}}, \ldots, Z_{k,N_{k,n}})
\]

The purpose of this paper is to give some answers to the following general question: "when does \(\{Z_{N_n}, n \geq 1\}\) converge weakly on \(\mathbb{R}^k\)?".

Under the fixed conditions on \(\{N_n, n \geq 1\}\) and \(\{Z_n, n \geq 1\}\), namely

(1) \( \min_i N_{i,n} \to \infty \) as \( n \to \infty \)

and

(2) \( Z_n \Rightarrow u \) as \( n \to \infty \)

we shall look for some additional conditions on indexed and indexing sequences under which

(3) \( \{Z_{N_n}, n \geq 1\} \) converges weakly on \(\mathbb{R}^k\) to measure \(u\).

(Here and in what follows "\( \overset{P}{\longrightarrow} \)", "\( \Rightarrow \)", "\( \Rightarrow \) a.s." denote "converges in probability", "converges weakly on \(\mathbb{R}^k\)" and "converges almost surely" respectively). In all our considerations in this paper we assume that (1) and (2) hold.

The obtained results can be divided into two main parts. In the first part (§3) are contained some answers to the three main questions:

I) Which condition must we put on \(\{Z_n, n \geq 1\}\) to enable us to conclude that (3) holds if we know nothing about \(\{N_n, n \geq 1\}\) (except condition (1))?
II) Which condition must we put on \( \{N_n, n \geq 1\} \) to enable us to conclude that (3) holds if we know nothing (except condition (2))?

III) Which common condition must we put on \( \{N_n, n \geq 1\} \) and \( \{Z_n, n \geq 1\} \) to enable us to conclude that (3) holds if we know only that (1) and (2) hold?

In the second part (§4) we discuss weak convergence of \( \{Z_n, n \geq 1\} \), where

\[
Z_n = \frac{(X_1 + X_2 + \ldots + X_n) - E(X_1 + X_2 + \ldots + X_n)}{\sigma^2(X_1 + X_2 + \ldots + X_n)}
\]

and \( \{X_n, n \geq 1\} \) is a sequence of independent r.v. with values in \( \mathbb{R}^k \).

Under the fixed assumption on \( \{Z_n, n \geq 1\} \), namely

(4) \[ Z_n \Rightarrow N(0, I) \text{ as } n \to \infty \]

we shall look for some additional conditions on \( \{X_n, n \geq 1\} \) and \( \{N_n, n \geq 1\} \) under which

\[
Z_n \Rightarrow N(0, I) \text{ as } n \to \infty
\]

(Here and in what follows \( N(0, I) \) stands for the standard Gaussian measure with mean zero and identity covariance matrix \( I \)).

The problem of centering and norming of sequence \( \{X_1 + X_2 + \ldots + X_n\} \), \( n \geq 1 \) is also considered. The obtained results are new even in the case \( k=1 \).

We ought to mention here that the problem discussed in this part in such a form has been considered only in [9] and later in [10]. In the case \( k=1 \) weak convergence of randomly indexed sequences have been studied in e.g. [1], [3], [5-9], [11] and [12]. In our considerations we shall use the idea related to those from [12].

REMARK. In this paper no assumption concerning the independence between indexed and indexing sequences is made. If they are independent and \( k=1 \) then (1) and (2) imply (3). In the case \( k \neq 1 \) in general this result is not true.

2. NOTATIONS AND DEFINITIONS.

Let \( a \in \mathbb{R}, x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathbb{R}^k \). We shall write

\[
(a) = (a, a, \ldots, a) \in \mathbb{R}^k, \|x\| = \max_i |x_i|
\]

\[
x \cdot y = (x_1 y_1, \ldots, x_k y_k), x/y = (x_1/y_1, \ldots, x_k/y_k)
\]

\[
x^a = (x_1^a, \ldots, x_k^a), x < y \iff x_i < y_i \text{ for all } i = 1, 2, \ldots, k
\]

If \( x < y \) then

\[
\langle x, y \rangle = \{z \in \mathbb{R}^k : x_i < z_i < y_i \text{ for all } i = 1, 2, \ldots, k \}
\]
If \( x_n = (x_{1,n}, \ldots, x_{k,n}) \), \( y_n = (y_{1,n}, \ldots, y_{k,n}) \in \mathbb{R}^k \), \( m_n = (m_{1,n}, \ldots, m_{k,n}) \in \mathbb{N}^k \) for \( n \geq 1 \) then
\[
\begin{align*}
x_n & \to \infty \iff \min_i x_{i,n} \to \infty \quad \text{as} \quad n \to \infty, \\
x_n & = o(y_n) \iff x_{i,n} = o(y_{i,n}) \quad \text{as} \quad n \to \infty, \quad i = 1, \ldots, k \\
x_{m_n} & = (x_{1,m_n}, \ldots, x_{k,m_n})
\end{align*}
\]

If \( x \in \mathbb{R}^k \), \( A \subset \mathbb{R}^k \) then
\[
x \cdot A = \{ z : z = x \cdot y, \ y \in A \}
\]

If \( X = (X_1, \ldots, X_k) \) r.v. then
\[
\mathbb{E}X = (\mathbb{E}X_1, \ldots, \mathbb{E}X_k), \quad \sigma^2 X = (\sigma^2 X_1, \ldots, \sigma^2 X_k)
\]

\( \mathcal{L}(X) \) is the law of r.v. \( X \).

We shall say that a set \( F \subset S \) is continuity set of measure \( u \) if \( u(F) = u(F^c) \).

By \( \beta_1 \) we denote the class of all sets \( x \in \mathbb{R}^k : x_1 < y_1, \ldots, x_k < y_k \), where \( y_i \in \mathbb{R} \) for all \( i = 1, 2, \ldots, k \) and \( \beta_2 \), the class of all convex Borel sets of \( \mathbb{R}^k \). If \( (S, d) \) is a polish space than by \( (S^k, d^\#) \) we denote the polish space with the distance defined as follows
\[
d^\#(x, y) = \max_i d(x_i, y_i)
\]
for any \( x = (x_1, \ldots, x_k), \ y = (y_1, \ldots, y_k) \in S^k \). In what follows we shall omit the star \( \# \).

If \( F \subset S \) then
\[
F^\beta = \{ y \in S : q(y, F) \leq \beta \}, \quad \text{where} \quad q(y, F) = \inf_{x \in F} d(y, x).
\]

CONVENTION. \( \sigma \)-field \( \mathcal{F} \) in \( \S 4 \) stands for \( \sigma \)-field generated by indexed r.v. So we shall have \( \mathcal{F} = \mathcal{F}(\mathcal{Z}_n, n \geq 1) \). \( (\Omega, \mathcal{F}, P) \) is nonatomic.

If \( D = D_1 \times D_2 \times \ldots \times D_k \subset \mathbb{N}^k \) then
\[
\min D = (\min D_1, \min D_2, \ldots, \min D_k), \quad \max D = (\max D_1, \max D_2, \ldots, \max D_k),
\]
where \( \min C = \min \{ k : k \in C \} \) for \( C \subset \mathbb{N} \).

For \( A \subset \mathcal{F} \) by \( P_A \) we denote the restriction measure defined as follows
\[
P_A(B) = P(A \cap B) \quad \text{for} \quad B \subset \mathcal{F}
\]

We shall say that a sequence \( A_1, A_2, \ldots, A_M \) is a partition of \( \Omega \) if for
\[
i \neq j \quad A_i \cap A_j = \emptyset, \quad \text{and} \quad \bigcup_{i=1}^M A_i = \Omega \quad \text{and} \quad A_i \in \mathcal{F} \quad \text{for} \quad i = 1, 2, \ldots, M.
\]

We have omitted the proofs of the results presented in \( \S 3 \). In fact they are based on [12] and Aldous' ones [1], which are applicable in this case, of course, after necessary modifications.
3. THREE MAIN QUESTIONS.

In this paragraph we answer the questions I), II), III).

THEOREM 1. If \( Z_n \to Z \text{ a.s. as } n \to \infty \) and \( \mathcal{L}(Z) = u \) then \( Z_n \to u \) as \( n \to \infty \) for all \( \{N_n, n \geq 1\} \) such that (1) holds.

THEOREM 1'. If \( \{Z_n, n \geq 1\} \) is such that "\( Z_n \to u \) as \( n \to \infty \) for all \( \{N_n, n \geq 1\} \) satisfying (1)" then there exist r.v. \( Z \) such that \( \mathcal{L}(Z) = u \) and \( Z_n \to Z \text{ a.s. as } n \to \infty \).

THEOREM 2. If \( \{N_n, n \geq 1\} \) is such that "\( N_n \to (a_n) = 0 \text{ as } n \to \infty \) for some sequence of natural numbers \( \{a_n, n \geq 1\} \)" then for all \( u \) and for all sequences of r.v. \( \{Z_n, n \geq 1\} \) such that \( Z_n \to u \) as \( n \to \infty \) \( Z_n \to u \) as \( n \to \infty \).

THEOREM 2'. Suppose that \( u \) is not concentrated at one point on \( S \). If a sequence \( \{N_n, n \geq 1\} \) is such that "\( Z_n \to u \) as \( n \to \infty \) for all sequences \( \{Z_n, n \geq 1\} \) such that \( Z_n \to u \) as \( n \to \infty \)" then there exists a sequence of natural numbers \( \{a_n, n \geq 1\} \) such that \( N_n = (a_n) \to 0 \text{ as } n \to \infty \).

We note there that we cannot weaken the assumptions of Theorems 1. and 2., for the Theorems 1. and 1' (2. and 2') respectively are in some sense invertible ones to them. Theorems 1. and 1' (2. and 2') answer the question I) (II) respectively.

In order to answer the question III) we shall develop of idea contained in [12]. In term of so called "introducing families" we shall give a condition of weak convergence of \( \{Z_n, n \geq 1\} \). One can easily check that the following lemma holds:

LEMMA 1. If \( \{N_n, n \geq 1\} \) satisfies (1) and for all \( n \geq 1 \) \( P \{N_n < \infty \} = 1 \) then there exists a family of subsets \( \{D_n = D_{1,n} \times D_{2,n} \times \ldots \times D_{k,n}, n \geq 1\} \) of \( \mathbb{N}^k \) such that

(i) \( P \{N_n \in D_n\} \to 1 \text{ as } n \to \infty \)
(ii) \( \min D_n \to \infty \text{ as } n \to \infty \)
(iii) \( D_n \) is finite subset of \( \mathbb{N}^k \) such that there exists a sequence of natural numbers \( \{k_n, n \geq 1\} \) such that \( k_n \to \infty \) as \( n \to \infty \) and \( (k_n) \subseteq D_n \) for \( n \geq 1 \).

REMARK. Throughout this paragraph without lost of generality we can and do assume that in condition (iii) we have \( k_n = n \) for \( n \geq 1 \).

DEFINITION 1. We shall say that a family of subsets \( \{D_n, n \geq 1\} \) of \( \mathbb{N}^k \) is "the introducing family to infinity" and denote \( D_n \in \text{I.F.} \) if she satisfies the conditions (ii) and (iii) of Lemma 1.
DEFINITION 2. Let $D_n \in I.F.$ and let $\varepsilon > 0$ and $\delta > 0$ be given. we shall say that $\{Z_n, n \geq 1\}$ satisfies "$(\varepsilon, \delta)$-Anscombe condition with respect to a family $\{D_n, n \geq 1\}$ (shortly $Z_n \in (\varepsilon, \delta)AC(D_n)$) if

$$\limsup_{n \to \infty} \Pr \left[ \max_{i \in D_n} d_i (Z_n) > \delta \right] \leq \delta$$

We shall say that $\{Z_n, n \geq 1\}$ satisfies Anscombe condition with respect to a family $\{D_n, n \geq 1\}$ (shortly $Z_n \in AC(D_n)$) if for all $\varepsilon > 0$ $Z_n \in (\varepsilon, \varepsilon)AC(D_n)$.

Let $\{D_n, n \geq 1\} \in I.F.$ be given.

THEOREM 3. The following conditions are equivalent:

(A) $Z_n \Rightarrow u$ as $n \to \infty$ and $Z_n \in AC(D_n)$

(B) $Z_n \Rightarrow u$ as $n \to \infty$ for all sequences $\{N_n, n \geq 1\}$ such that $\Pr[N_n \in D_n] \to 1$ as $n \to \infty$.

COROLLARY 1. $Z_n \Rightarrow u$ as $n \to \infty$ if there exists a family $D_n \in I.F.$ such that $\Pr[N_n \in D_n] \to 1$ as $n \to \infty$ and $Z_n \in AC(D_n)$.

COROLLARY 2. $Z_n \Rightarrow u$ as $n \to \infty$ if for all $\beta > 0$ there exists $D_n(\beta) \in I.F.$ such that $\liminf_{n \to \infty} \Pr[N_n \in D_n(\beta)] \geq 1 - \beta$ and $Z_n \in (\beta, \beta)AC(D_n(\beta)).$

Let us note at the end of this paragraph the following lemma:

LEMMA 2. If $Z_i \in (\beta, \beta)AC(D_i, n \times D_i, n \times \cdots \times D_i, n)$ then $Z_i,n \in (\beta, \beta)AC(D_i,n)$ for all $i = 1, 2, \ldots, k$.

If $Z_i,n \in (\beta, \beta/k)AC(D_i,n)$ for all $i = 1, 2, \ldots, k$ then $Z_n \in (\beta, \beta)AC(D_1,n \times D_2,n \times \cdots \times D_k,n)$.

4. SOME CENTRAL LIMIT THEOREMS.

I. This part is of purely technical character. We shall note there some facts which characterise "weak mixing convergence" (Proposition 1) and proof some general theorem (Theorem 4.) from which we shall deduce some random limit theorems for random sums of independent random vectors (Theorems 5, 6, 7, 8).

One can check that the following proposition holds:
PROPOSITION 1. The following conditions are equivalent

A) \( Z_n \Rightarrow u \text{ mixing as } n \to \infty \)
B) For all fixed \( \mathcal{F} \)-measurable r.v. \( Y(Z_n,Y) \Rightarrow u \otimes v \text{ as } n \to \infty \) where \( v = \mathcal{L}(Y) \)
C) For all fixed \( A \in \mathcal{F} \) such that \( P \{ |A| > 0 \} \) and all fixed closed set \( \mathcal{F} \in S^k \)
\[ \limsup_{n \to \infty} P \{ Z_n \in \mathcal{F} \mid A \} \leq u \{ \mathcal{F} \} \]
(Compare Proposition 2 [2])

In all our considerations in this part we assume that

\[ \min_{i=1}^{N_i,n} \frac{P}{P} \Rightarrow \infty \text{ as } n \to \infty \]

and

\[ Z_n \Rightarrow u \text{ mixing as } n \to \infty \]

THEOREM 4. If for all \( \beta > 0 \) there exists a partition \( A_1, A_2, \ldots, A_k \)
of \( \Omega \), the families \( D_1^n(\beta), D_2^n(\beta), \ldots, D_k^n(\beta) \in \mathcal{F} \) and the sequences of natural numbers \( \{i(n), n \geq 1 \} \) \( i = 1, 2, \ldots, k \) such that \( i(n) \to \infty \) as \( n \to \infty \) for all \( i = 1, 2, \ldots, k \)

\begin{align*}
\limsup_{n \to \infty} \sum_{i=1}^{k^B} P_{A_i} \left[ \max_{k \in D_n^i(\beta)} d(Z_k, Z_i(n)) > \beta \right] & \leq \beta \\
\text{and} \\
\limsup_{n \to \infty} \sum_{i=1}^{k^B} P_{A_i} \left[ N_n \not\in D_n^i(\beta) \right] & \leq \beta
\end{align*}

then \( Z_n \Rightarrow u \text{ mixing as } n \to \infty \).

PROOF. Let \( \beta > 0 \) and closed set \( \mathcal{F} \in S^k \) be given. Choosing \( A_1, \{D_n^i(\beta), n \geq 1 \}, \{i(n), n \geq 1 \}, i = 1, 2, \ldots, k \) as in assumptions we obtain

\[ P \{ Z_n \in \mathcal{F} \} \leq \sum_{i=1}^{k^B} P_{A_i} \left[ Z_n \in \mathcal{F}, \max_{k \in D_n^i(\beta)} d(Z_k, Z_i(n)) < \beta, N_n \in D_n^i(\beta) \right] + \sum_{i=1}^{k^B} P_{A_i} \left[ N_n \not\in D_n^i(\beta) \right] \]

\[ + \sum_{i=1}^{k^B} P_{A_i} \left[ Z_i(n) < F^\beta \right] \]

Thus by (5), (6) and C) of Proposition 1, we get
\[ \limsup_{n \to \infty} P \left[ Z_n \in F \right] \leq u(F^2) + 2\beta \]

Since \( \beta > 0 \) can be chosen arbitrarily small, the last inequality, Theorem 2.1 [5] and Aldous' Remark 3 [1] end the proof of Theorem 4.

II. Let \( \{X_n, n \geq 1\} \) be a sequence of independent \( \mathbb{R}^k \)-valued r.v.

Define
\[
S_n = \sum_{i=1}^{n} X_i, \quad A_n = ES_n, \quad s_n^2 = \sigma^2 S_n
\]

Throughout this part we assume that

(4) \[ Z_n = (S_n - A_n)/s_n \Rightarrow N(0, I), \quad s_n \to \infty \text{ as } n \to \infty \]

Of course, from [10], \( Z_n \Rightarrow N(0, I) \) mixing as \( n \to \infty \).

CASE \( k = 1 \).

THEOREM 5. If \( \{N_n, n \geq 1\} \) is such that there exists a sequence of real numbers \( \{a_n, n \geq 1\} \), such that \( s_n \to \infty \) as \( n \to \infty \) and
\[
s_n^2/a_n \stackrel{P}{\to} \lambda \text{ as } n \to \infty
\]

for some real valued r.v. \( \lambda \) such that \( P \left[ 0 < \lambda < \infty \right] = 1 \), then
\[ Z_n \Rightarrow N(0, I) \text{ as } n \to \infty. \]

PROOF. Without loss of generality we can and do assume that \( A_n = 0 \).

We shall show that \( \{N_n, n \geq 1\} \) and \( \{Z_n, n \geq 1\} \) satisfy the assumptions of Theorem 4. Let \( \beta > 0 \) be given. We choose \( C > c > 0 \) such that
\[ P \left[ \lambda \in (c, C) \right] \geq 1 - \beta/2. \]

Let \( c = x_0 < x_1 < x_2 < \ldots < x_k = C \) are such that \( \delta = \max_i (x_i - x_{i-1}) \leq \delta \leq c/64. \)

We can and do assume that \( \{x_i, i=0,1,2,\ldots,k\} \) are continuity points of distribution function of \( \lambda \). Let us define
\[
A_0 = [\lambda \notin (c,C)], \quad A_i = [\lambda \notin (x_{i-1},x_i)] \text{ for } i = 1,2,\ldots,k
\]
\[
D_0 = \{n\}, \quad D_i^n(B) = \{k: x_{i-1} \leq x_k < x_i \text{ for } i = 1,2,\ldots,k
\]

One can easy check that for any \( \delta > 0 \) \( \{N_n, n \geq 1\} \) satisfies (6) with such defined \( \{A_i, \text{ i}=0,1,2,\ldots,k\} \) and \( \{D_i^n, n \geq 1\} \) \( i = 0,1,2,\ldots,k \) (see e.g. [81]). On the other hand, by Lemma 3 [6], for any \( A \in \mathcal{F} \) we have
\[
\limsup_{n \to \infty} P_A \left[ \max_{k \in D_i^n(B)} d(Z_k, Z_n) \geq \beta \right] \geq P \left[ A \right] \limsup_{n \to \infty} \max_{k \in D_i^n(B)} d(Z_k, Z_n) \geq \beta
\]
So, in order to check (5) with \( i(n) = n \) for \( i = 0, 1, 2, \ldots, k \) it is enough to show that
\[
\lim sup_{n \to \infty} P \left[ \max_{k \in D_n^i} d(Z_k, Z_n) \geq \beta \right] < \beta
\]
Putting \( n = \min D_n^i(\beta) \), \( \overline{n} = \max D_n^i(\beta) \) and omitting \( \beta \) in \( D_n^i(\beta) \) we have
\[
I_n = P \left[ \max_{k \in D_n^i} |S_k/s_k - S_n/s_n| \geq \beta \right] \leq 2P \left[ \max_{k \in D_n^i} |S_k/s_k - \overline{s}_n| \geq \beta/2 \right] \leq 2P \left[ \max_{k \in D_n^i} |S_k - \overline{S}_n| \geq \overline{s}_n \beta/4 \right] + 2P \left[ \max_{k \in D_n^i} |(1/s_n - 1/s_k) \geq \beta/4 \right] = 2I_n^1 + 2I_n^2
\]
Using the classical Kolmogorov's inequality we obtain
\[
2I_n^1 \leq 32(x_i - x_{i-1})/x_{i-1}^2 \beta^2 \leq \beta/2
\]
On the other hand for \( k \in D_n^i(\beta) \) we have
\[
|1/s_n - 1/s_k| \leq (1/s_n - 1/s_{\overline{n}})
\]
Using than again Kolmogorov's inequality we obtain
\[
2I_n^2 \leq 32(x_i - x_{i-1})/x_{i-1}^2 \beta^2 \leq \beta/2
\]
Proof of Theorem 5. is finished.

REMARK. From our Theorem 5. we can obtain the results of papers [3], [7].

CASE \( k \neq 1 \).

THEOREM 6. The following conditions are equivalent:

(A) \( Z_n \Rightarrow N(0,1) \) mixing as \( n \to \infty \)

(B) \( Z_n \Rightarrow N(0,1) \) mixing as \( n \to \infty \) for all \( \{N_n, n \geq 1\} \) satisfying
\[
\frac{s_n^2}{s_k^2} \Rightarrow (\lambda) \quad \text{as} \quad n \to \infty
\]
for some real-valued r.v. \( \lambda \) such that \( P \left[ 0 < \lambda < \infty \right] = 1 \).

PROOF. The implication \((B) \Rightarrow (A)\) is obvious. In order to proof that \((A) \Rightarrow (B)\) we shall show that \( \{N_n, n \geq 1\} \) and \( \{Z_n, n \geq 1\} \) satisfy
the assumptions of Theorem 4. Let $\beta > 0$ be given. We choose $c > c > 0$ such that

$$P \{ \lambda \in (c, C) \} \geq 1 - \beta/2.$$  

Let $c = x_0 < x_1 < x_2 < \ldots < x_n = C$ are continuity points of measure $\mathcal{L}(\lambda)$ such that $\max(x_i - x_{i-1}) \leq \beta^3 c/64k$. Let us define

$$A_0 = \{ \lambda \notin (c, C) \}, \quad A_i = \{ \lambda \in (x_{i-1}, x_i) \} \quad \text{for } i = 1, 2, \ldots, k,$$

$$D^0_n(\beta) = \{(n)\}, \quad D^i_n(\beta) = \{k: (x_{i-1}) \leq s_k^2/s_n^2 < (x_i)\} \quad \text{for } i = 1, 2, \ldots, k,$$

$$i(n) = n \text{ for } i = 0, 1, 2, \ldots, k.$$

It is easy to check (6) (see e.g. [8]). To check (5), let us observe that by Lemma 3 [6], Lemma 2 and Theorem 5., for $i = 1, 2, \ldots, k$ we get

$$\limsup_{n \to \infty} P_{A_i} \{ \max_{k \in D^i_n(\beta)} d(Z_k, Z_n) \geq \beta \} = P \{ A_i \limsup_{n \to \infty} P \{ \max_{k \in D^i_n(\beta)} d(Z_k, Z_n) \geq \beta \} \leq P \{ A_i \} 64k(x_i - x_{i-1})/x_{i-1}^2 \leq \beta P \{ A_i \}$$

That ends the proof.

III. Assume that $\{N_n = (N_1, N_2, \ldots, N_k), n \geq 1\}$ is such that

$$(7) \quad (A_{n_n} - a_n)/b_n \xrightarrow{P} X \text{ as } n \to \infty$$

and

$$(8) \quad (s_{n_n}^2 - s_n^2)/c_n^2 \xrightarrow{P} (Y) \text{ as } n \to \infty$$

for some sequences $\{a_n \in \mathbb{R}^k, n \geq 1\}, \{b_n \in \mathbb{R}^k, n \geq 1\}, \{c_n \in \mathbb{R}^k, n \geq 1\}$ and r.v. $X$ and $Y$ with values in $\mathbb{R}^k$ and $\mathbb{R}$ respectively. Putting $\mathcal{L}(X) = \mathbb{I}$ we obtain

THEOREM 7. If (4), (7) and (8) hold and $c_n^2 = o(s_n^2), b_n = O(c_n)$ as $n \to \infty$, then for every $C \in B_i, i = 1, 2$

$$(9) \quad \lim_{n \to \infty} P \{ (s_{n_n}^2 - a_n)/s_n \in C \} = u_n \ast N(0, I) \{ C \} = 0$$

where $u_n \{ C \} = u \{ s_n \cdot b_n^{-1} \cdot C \}$, and

$$(10) \quad Z_{N_n} \Rightarrow N(0, I) \text{ as } n \to \infty.$$  

REMARK. Conclusion (10) holds even without assumptions "(7)" and "$b_n = O(c_n)$ as $n \to \infty"."
PROOF. We can proof (9) of Theorem 7. repeating proof of Theorem 1. [11]. Theorem 7. differs from Theorem 1. [11] in the lack of assumption "\{X_n, n \geq 1\} has independent components", which is replaced here by special kind "centering" and "norming" vectors of \{s^2_n, n \geq 1\} and special kind of limit r.v. in (8). In the proof of Theorem 1 [11] the assumption in inverted commas has only been needed in conclusion (10). Namely, we wanted to know if (see conclusion (10) [11])

\[(10') \quad \frac{(S_{nj} - A_{nj})/s^2_{nj}}{N} = N(0, I) \text{ mixing as } n \to \infty \]

for some sequence \{n_j, n \geq 1\} defined as follows: if \(x_j \in R\) then

\[ n_j = \min \{i: (x_j) \leq (s^2_i - s^2_n)/c \leq (x_j + B)\} =: \min D^j_n(B) \]

for some \(B > 0\) sufficiently small. But by assumption \(c^2_n = o(s^2_n)\) we get

\[ s^2_{nj}/s^2_n \to (1) \text{ as } n \to \infty \]

for any \(x_j \in R\) and any \(B > 0\). Then by Theorem 6 (10') holds.

In order to proof (10) let us observe that for given \(B > 0\) our sequences \{N_n, n \geq 1\} and \{Z_n, n \geq 1\} satisfy assumptions of Corollary 2 with \(D_n(\beta)\) defined as follows

\[ D_n(\beta) = \{i: (\beta) \leq (s^2_i - s^2_n)/c \leq (\beta)\} \]

where \(\beta > 0\) is such that \(P \{ -\beta < Y < \beta \} \geq 1 - \beta\). (Checking this is based on the estimations given in the proof of Theorem 5, and we omit it).

THEOREM 8. If (8) holds for r.v. Y such that \(P \{ 0 < Y < \infty \} = 1\) and \(c^2_n = o(s^2_n)\) as \(n \to \infty\) then \(Z_n \to N(0, I)\) as \(n \to \infty\).

PROOF. Let us observe that for given \(B > 0\) \{N_n, n \geq 1\} and \{Z_n, n \geq 1\} satisfy (5) and (6) of theorem 4. with the following sequences

\[ A_0 = \{ Y \not\in (c, C) \} \quad A_i = \{ Y \in (x_{i-1}, x_i) \} \quad \text{for } i = 1, 2, \ldots, k_B \]

\[ D^0_n(\beta) = \{n\} \quad D^i_n(\beta) = \{i: (x_{i-1}) \leq (s^2_i - s^2_n)/c_n \leq (x_i)\} \quad i = 1, 2, \ldots, k_B \]

\[ i(n) = n \text{ for } i = 1, 2, \ldots, k_B + 1 \]

where \(c = x_0 < x_1 < x_2 < \ldots < x_k = C\) are continuity points of r.v. Y such that \(P \{ Y \in (c, C) \} \geq 1 - B^2/2\beta^2\) and \(\max (x_i - x_{i-1}) \leq 8\beta/64\), where \(D = \max \lim sup c^2_i, s^2_i, n \to \infty\). Checking of this fact is based on the estimations given in the proof of Theorem 5, and we leave to the reader.
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ON THE RATE OF CONVERGENCE FOR DISTRIBUTIONS OF INTEGRAL TYPE FUNCTIONALS

Z. Rychlik and I. Szyszkowski

1. Introduction. Let \( \{(S_{nk}, F_{nk}) \} \) be a square-integrable martingale for each \( n \geq 1 \), and define

\[ X_{nk} = S_{nk} - S_{nk-1} \quad \text{and} \quad \theta_{ni}^2 = \frac{X_{ni}^2}{v_{nk}^2}, \]

where \( F_{n0} \) is a \( G \)-field such that \( F_{n0} \leq F_{n1} \). \( \{(S_{nk}, F_{nk}) \} \) is called a triangular martingale array. Suppose that each martingale has been normalised so that \( E S_{nk}^2 = E V_{nk}^2 = 1 \) and \( k_n \to \infty \) as \( n \to \infty \).

Let us put

\[ b_{ni}^2 = \theta_{ni}^2 v_n^2 , \quad b_{ni} = \frac{b_{ni}^2}{v_n^2} , \quad B_{ni}^2 = \sum_{k=1}^{k_n} b_{ni}^2 , \quad B_{n0}^2 = v_{n0}^2 = 0 , \]

1 \( \leq i \leq k_n \), \( n \geq 1 \).

In what follows we assume that

\[ \mathbb{L}_n^{(s)} = \sum_{k=1}^{k_n} E |X_{nk}|^{2s} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for some fixed} \quad s > 1 . \]

Let \( f(t,x) \) be a continuous function which is defined and has continuous partial derivatives on the set \([0,1] \times \mathbb{R}\), where \( \mathbb{R} \) denotes the set of real numbers. We assume that there exist positive constants \( \alpha \) and \( \mathcal{L} \) such that

\[ |Df(t,x)| \leq \mathcal{L} (1 + |x|^{\alpha}) , \quad (t,x) \in [0,1] \times \mathbb{R} , \]

where \( D \) denotes either the identity operator \( I \) or partial deriv-
Let $X_n(t), 0 \leq t \leq 1$, be the random function defined as follows

$$X_n(t) = S_n + X_{nk}(t - b_{nk}^2/k^n) / b_{nk}^2,$$

for $0 \leq k \leq k_n, n \geq 1$.

It is well known (cf. [1], [4] or [10]) that if $V_n^2 \xrightarrow{P} 1$ and $L_n^{(s)} \rightarrow 0$ as $n \rightarrow \infty$, for some positive $s > 1$, then under the assumptions given above $X_n \xrightarrow{D} W$, as $n \rightarrow \infty$, where $W$ is the Wiener measure on $C[0,1]$ with the corresponding Wiener process $\{W(t), 0 \leq t \leq 1\}$, (cf. [2], Sec. 9). In what follows $W = \{W(t), 0 \leq t \leq 1\}$. Hence, if $\Phi$ is a continuous functional defined on $C[0,1]$, then (cf. [2, p. 30])

$$\Phi(X_n) \xrightarrow{D} \Phi(W) \quad \text{as} \quad n \rightarrow \infty.$$ 

2. The convergence rate for distributions of integral type functionals

From now on we assume that the probability space on which the random variables $X_{ni}, 1 \leq i \leq k_n$, are defined is reach enough so that one can define on it the Wiener process.

The proofs of the main results are based on a martingale form of the Skorokhod representation theorem which we state as a lemma in the interests of clarity. This lemma is presented in [7] and [12].
Lemma 1. (Strassen [12], Theorem 4.3)

Let \( \{X_n, n \geq 1\} \) be a sequence of random variables such that for all \( n \geq 1 \), \( \mathbb{E}(X_n^2 | X_{n-1}, \ldots, X_1) \) is defined and \( \mathbb{E}(X_n | X_{n-1}, \ldots, X_1) = 0 \) a.s. Then there is a Brownian motion \( \{W(t), t \geq 0\} \) together with a sequence of nonnegative random variables \( \{\tau_n, n \geq 1\} \) such that, for all \( n \geq 1 \),
\[
\sum_{k=1}^{n} X_k = W\left(\sum_{k=1}^{n} \tau_k\right) \quad \text{a.s.}
\]
Moreover, if \( F_n \) is the \( \sigma \)-field generated by \( X_1, X_2, \ldots, X_n \) and \( \tau(t) \) for \( 0 \leq t \leq \sum_{i=1}^{n} \tau_i \), then \( \tau_n \) is \( F_n \)-measurable, \( \mathbb{E}(\tau_n | F_{n-1}) \) is defined and
\[
\mathbb{E}(\tau_n | F_{n-1}) = \mathbb{E}(X_n^2 | F_{n-1}) = \mathbb{E}(X_n^2 | X_{n-1}, \ldots, X_1) \quad \text{a.s.}
\]
If \( k \) is a real number, \( k \geq 1 \), and \( \mathbb{E}(|X_n^{2k} | X_{n-1}, \ldots, X_1) \) is defined, then \( \mathbb{E}(\tau_n^k | F_{n-1}) \) is also defined and
\[
\mathbb{E}(\tau_n^k | F_{n-1}) \leq L_k \mathbb{E}(X_n^{2k} | F_{n-1}) = L_k \mathbb{E}(|X_n^{2k} | X_{n-1}, \ldots, X_1),
\]
a.s., where \( L_k \) is a constant which depends only on \( k \).

In what follows \( K_s, f \) denotes a positive constant which depends only on \( s \) and the function \( f \) and the same symbol may be used for different constants.

Theorem 1. Let \( \{(S_{nk}, F_{nk}), 1 \leq k \leq k_n, n \geq 1\} \) be a triangular martingale array such that \( \mathbb{E}S_{nk}^2 = \mathbb{E}V_n^2 = 1 \) and satisfying (1) with some fixed \( s > 1 \). Assume \( \Phi \) is a functional defined by (4) and such that the distribution of the random variable \( \Phi(W) \) satisfies the Lipschitz condition. Then
\[
\sup_x |\mathbb{P}(F \leq x) - \mathbb{P}(\Phi(W) \leq x)| \leq
\]
\[
\leq K_{s, f} \left( \log R_{ns}^{-1} \right)^{s/2} \min\left( 1/(1+s), 1/(s-1) \right) \left( \log R_{ns} \right)^{s/4},
\]
where \( R_{ns} = \sum_{k=0}^{k_n-1} \mathbb{E}(V_{nk}^2 - 1)^{s} \) and \( F = \sum_{k=0}^{k_n-1} f(B_{nk}^2, S_{nk})b_n^2, k+1 \).
One can easily note that if $1 < s < 5/3$ and $X_{ni}, 1 \leq i \leq k_n$, are independent random variables then Theorem 1 gives Theorem 1 of [3].

Let us put

$$\Phi(X_n) = \int_0^1 f(t,X_n(t)) \, dt.$$

**Theorem 2.** Suppose the assumptions of Theorem 1 hold. Then in (5) we can put $\Phi(X_n)$ instead of $F$.

3. **Proof of Theorem 1.** Applying Lemma 1 to the sequence of random variables $\{X_{ni}, 1 \leq i \leq k_n\}, n \geq 1$, we note that there exists a Brownian motion $\{W(t), t \geq 0\}$ and nonnegative random variables $\tau_{n1}, \tau_{n2}, \ldots, \tau_{nk_n}$ such that with probability one the following equality

$$F = \sum_{k=1}^{k_n-1} f(B_{nk}, S_{nk}) b_{n,k+1}^2 = \sum_{k=0}^{k_n-1} f(B_{nk}, W(T_{nk})) b_{n,k+1}^2 = F_1$$

holds, where $T_{nk} = \sum_{i=1}^{k} \tau_{ni}$, $k = 1, 2, \ldots, k_n$, $T_{n0} = 0$.

One can observe that for every real number $x$ and any $\delta > 0$

$$P( |F_1 - \Phi(W)| > \delta ) \geq P( F_1 \leq x, \Phi(W) > x + \delta ) \geq$$

$$\geq P( F_1 \leq x ) - P( \Phi(W) \leq x ) - P( x < \Phi(W) \leq x + \delta ).$$

Similarly

$$P( |F_1 - \Phi(W)| > \delta ) \geq P( \Phi(W) < x - \delta ) -$$

$$- P( F_1 \leq x ) = P( \Phi(W) \leq x ) - P( F_1 \leq x ) - P( x - \delta \leq \Phi(W) \leq x ).$$

Taking into account the inequalities given above and the assumption on the distribution function $\Phi(W)$, we get
\[ \sup_x |P(F_1 \leq x) - P(\overline{F}(W) \leq x)| \leq P(|F_1 - \overline{F}(W)| > \delta) + \\
+ P(x - \delta \leq \overline{F}(W) \leq x + \delta) \leq P(|F_1 - \overline{F}(W)| > \delta) + \\
+ 2L\delta, \]

where \( L \) is a positive constant.

It is easy to see that

\[ P(|F_1 - \overline{F}(W)| > 4\delta) \leq P(|F_1 - \overline{F}(W)| > \delta), \]

\[ \max_{0 \leq t \leq 2} |W(t)| \leq A, T_{nk} \leq 2 \] + \( P(\max_{0 \leq t \leq 2} |W(t)| > A) \) +

\[ + P( T_{nk} > 2 ). \]

Let us put \( B = \bigcap_{0 \leq t \leq 2} \max_{0 \leq t \leq 2} |W(t)| \leq A, T_{nk} \leq 2 \). Then on the set \( B \), by (2), we have

\[ |Df(B_{nk}^2, W(T_{nk}))| \leq \Omega \left( 1 + \frac{1}{A^2} \right), \]

where \( D \) stands, as in above, for the operators \( \partial/\partial t \) and \( \partial/\partial x \) as well as for the identity operator \( I \).

Define \( Df^A(t,x) = Df(t,x) \) if \( |x| \leq A \) and \( Df^A(t,x) = 0 \) if \( |x| > A \). Now, by the mean value theorem, we get

\[ |\overline{F}(W) - F_1| I(B) = \int_0^1 f(t,W(t)) dt - \\
- \sum_{k=0}^{k-1} f(B_{nk}^2, W(T_{nk})) b_{n,k+1}^2 I(B) \leq \overline{\Phi}_1 + \overline{\Phi}_2 + \overline{\Phi}_3 + \overline{\Phi}_4, \]

where

\[ \overline{\Phi}_1 = \sum_{k=0}^{k-1} f^A(B_{nk}^2, W(T_{nk}))(\tau_{n,k+1} - b_{n,k+1}^2) I(B), \]
\( \Phi_2 = \left| \int_{T_{n_k}}^1 f^A(t, W(t)) \, dt \right| I(B), \)

\( \Phi_3 = \left| \sum_{k=0}^{k_n-1} \int_{T_{nk}}^{T_{n_k,k+1}} \frac{\partial f^A}{\partial t} (B_{nk}^2 + Q_{nk}(s), W(T_{nk}) + \right. \)

\( + \left. \psi_{nk}(s) \right| W(s) - W(T_{nk}) \right| ds \left| I(B) \right|, \)

\( \Phi_4 = \left| \sum_{k=0}^{k_n-1} \int_{T_{nk}}^{T_{n_k,k+1}} \frac{\partial f^A}{\partial t} (B_{nk}^2 + Q_{nk}(s), W(T_{nk}) + \right. \)

\( + \left. \psi_{nk}(s) \right| W(s) - W(T_{nk}) \right| ds \left| I(B) \right|, \quad 1 \leq k \leq k_n, \quad n \geq 1, \)

and the functions \( Q_{nk} \) and \( \psi_{nk} \) are such that, for every \( T_{nk}, \)

\( s \leq T_{n,k+1}, \quad \left| Q_{nk}(s) \right| \leq \left| B_{nk}^2 - s \right|, \) and

\( \left| \psi_{nk}(s) \right| \leq \left| W(s) - W(T_{nk}) \right|. \) Hence by (8)

\[ (9) \quad P\left( \left| E_1 - \Phi(W) \right| > 4 \delta, \quad B \right) \leq \sum_{k=1}^{4} P\left( \Phi_k > \delta \right). \]

It is well known that

\[ (10) \quad P\left( \max_{0 \leq t \leq 2} \mid W(t) \mid > A \right) \leq 4 \exp(-A^2/4). \]

Now we are going to estimate the probabilities \( P\left( \Phi_k > \delta \right), \)

\( 1 \leq k \leq 4. \) Some obtained estimations we present in Lemmas in the interests of clarity.

At first we assume that \( 1 \leq s \leq 2. \) Then, by the well known inequality for martingales, for every \( x > 0, \) we have
Thus, by (11), we have

\[
\mathbb{P}\left( \max_{k \leq k_n} \left| T_{nk} - V_{nk}^2 \right| > x \right) \leq x^{-s} \mathbb{E} \left| T_{nk_n} - V_{n}^2 \right|^s.
\]

Therefore, by Burkholder's martingale extension of the Marcinkiewicz-Zygmund inequality (Theorem 9 [5] or Lemma 6) there exists a finite universal constant \( M_s \) such that

\[
\mathbb{E} \left| T_{nk_n} - V_{n}^2 \right|^s \leq M_s \mathbb{E} \left( \sum_{k=1}^{k_n} \left( \tau_{nk} - \sigma_{nk}^2 \right)^2 \right)^{s/2} \leq
\]

\[
\leq M_s \mathbb{E} \left( \sum_{k=1}^{k_n} \left| \tau_{nk} - \sigma_{nk}^2 \right|^s \right), \quad \text{since } s/2 \leq 1,
\]

\[
\leq 2^{s-1} M_s \sum_{k=1}^{k_n} \left( \mathbb{E} \tau_{nk}^s + \mathbb{E} \sigma_{nk}^{2s} \right) \leq
\]

\[
\leq 2^{s-1} M_s \sum_{k=1}^{k_n} \left( \mathbb{E} |X_{nk}|^{2s} + \mathbb{E} \sigma_{nk}^{2s} \right),
\]

using Lemma 1 in the last step. Thus using the inequality

\[
\mathbb{E} \sigma_{nk}^{2s} = \mathbb{E} \left( \mathbb{E} (X_{nk}^2 | P_{n,k-1}) \right)^s \leq \mathbb{E} |X_{nk}|^{2s},
\]

we get
(13) $P(\max_{k \leq k_n} |T_{nk} - E_{nk}^2| > qR_{ns}^{1/(1+s)}) \leq K_{s,q}R_{ns}^{1/(1+s)}$, 

where $1 < s \leq 2$. Furthermore, by (10), we get

(14) $P(\max_{0 \leq t \leq 2} |W(t)| > 2(\log R_{ns}^{-1})^{1/2}) \leq 4R_{ns}$.

From this point on we will assume that $R_{ns} < e^{-1}$. Thus, by (2), on the set $B = \{\max_{0 \leq t \leq 2} |W(t)| \leq A, T_{nk} \leq 2\}$ with $A = 2(\log R_{ns}^{-1})^{1/2}$ we obtain

(15) $\max_{k \leq k_n} |D^2R_{nk}^2, W(T_{nk})| \leq \Omega_0(\log R_{ns}^{-1})^{2/2}$,

where $\Omega_0$ is a constant which depends only on the function $f(t,x)$.

Now by (13) and (15) we obtain

(16) $P(\max_{0 \leq t \leq 2} |T_{nk} - 1| > qR_{ns}^{1/(1+s)}) \leq K_{s,q}R_{ns}^{1/(1+s)}$, $1 < s \leq 2$.

and, for every $1 < s \leq 2$,

(17) $P(T_{nk} > 2) \leq P(\max_{k \leq k_n} |T_{nk} - 1| > 1) \leq K_{s,q}R_{ns}^{1/(1+s)}$.

**Lemma 2.** If $1 < s \leq 2$, then

(18) $P(\max_{0 \leq t \leq 2} |T_{nk} - E_{nk}^2| > q(\log R_{ns}^{-1})^{2/2}) \leq K_{s,q,f}R_{ns}^{1/(1+s)}$,

where $K_{s,q,f}$ denotes a constant which depends only on $s$, $q$ and the function $f$. 
Proof. Let $Z_{nk} = Y_{n0} + Y_{n1} + \cdots + Y_{nk}$, where

$$Y_{ni} = f^A(v_{n,i-1}^2, W(T_{ni-1}))(\mathcal{C}_{n,i} - \sigma_{n,i}^2), \quad 1 \leq i \leq k_n.$$ 

Then \( \{ (Z_{nk}, G_{nk}), 1 \leq k \leq k_n, n \geq 1 \} \) is a triangular martingale array. On the other hand

\begin{align*}
(19) \quad & f^A(B_{ni}^2, W(T_{ni}))(\mathcal{C}_{n,i+1} - \sigma_{n,i+1}^2) = Y_{n,i+1} + \\
& + [f^A(B_{ni}^2, W(T_{ni}^2)) - f^A(v_{ni}^2, W(T_{ni}^2))] \mathcal{C}_{n,i+1} + \\
& + f^A(B_{ni}^2, W(T_{ni}^2)) \sigma_{n,i+1}^2 (v_{n}^2 - 1)/v_n^2 + f^A(v_{ni}^2, W(T_{ni}^2)) - f^A(B_{ni}^2, W(T_{ni}^2)) \sigma_{n,i+1}^2.
\end{align*}

But putting $A = 2(\log R_n^{-1})^{1/2}$ and using (15) on the set $B$ (defined above) we get

\begin{align*}
(20) \quad & \left| \sum_{i=1}^{k_n-1} [f^A(B_{ni}^2, W(T_{ni}^2)) - f^A(v_{ni}^2, W(T_{ni}^2))] \mathcal{C}_{n,i+1} \right| \\
& \leq 2 \Omega_o(\log R_n^{-1})^{1/2} |v_n^2 - 1|,
\end{align*}

\begin{align*}
(21) \quad & \left| \sum_{i=1}^{k_n-1} f^A(B_{ni}^2, W(T_{ni}^2)) \sigma_{n,i+1}^2 (v_{n}^2 - 1)/v_n^2 \right| \\
& \leq \Omega_o(\log R_n^{-1})^{1/2} |v_n^2 - 1|,
\end{align*}

and

\begin{align*}
(22) \quad & \left| \sum_{i=1}^{k_n-1} [f^A(v_{ni}^2, W(T_{ni}^2)) - f^A(B_{ni}^2, W(T_{ni}^2))] \sigma_{n,i+1}^2 \right| \\
& \leq \Omega_o(\log R_n^{-1})^{1/2} v_n^2 |v_n^2 - 1| \leq \Omega_o(\log R_n^{-1})^{1/2} (|v_n^2 - 1|^2 +
\end{align*}
Thus, taking into account (19) - (22), we get

\[ P\left( \frac{1}{1} > q \left( \log R_{ns}^{-1} \right)^{\alpha/2} R_{ns}^{1/(1+s)} \right) \leq \]

\[ \leq P\left( |Z_{nk}| > q \left( \log R_{ns}^{-1} \right)^{\alpha/2} R_{ns}^{1/(1+s)} / 4 \right) + (32)^{s/2} E |V_n^2 - 1|^{s/2}(1+s) \leq \]

\[ \leq P\left( |Z_{nk}| > q \left( \log R_{ns}^{-1} \right)^{\alpha/2} R_{ns}^{1/(1+s)} / 4 \right) + K_{s,q,f} R_{ns}^{1/(1+s)} \]

(since \( R_{ns} < e^{-1} < 1 \)), where \( K_{s,q,f} \) is a constant depending only on \( s, q \) and the function \( f \).

On the other hand, if \( 1 < s \leq 2 \), then by Markov's and Burkholder's inequalities (\( C_5 \) or \( C_7 \)) and (15) we obtain

\[ P\left( |Z_{nk}| > q \left( \log R_{ns}^{-1} \right)^{\alpha/2} R_{ns}^{1/(1+s)} \right) \leq \]

\[ \leq M_s E\left( \sum_{k=0}^{k_n} Y_{nk}^2 \right)^{s/2} q^s \left( \log R_{ns}^{-1} \right)^{\alpha s/2} R_{ns}^{s/(1+s)} \leq \]

\[ \leq M_s \Omega^{s/2} E\left( \sum_{k=1}^{k_n} \left( \zeta_{nk} - 6_{nk}^2 \right)^2 \right)^{s/2} / q^s R_{ns}^{s/(1+s)} \leq \]

\[ \leq M_s \Omega^{s/2} \left( \sum_{k=1}^{k_n} \left| \zeta_{nk} - 6_{nk}^2 \right|^s \right) / q^s R_{ns}^{s/(1+s)} \leq \]

\[ \leq 2^{s-1} M_s \Omega^{s/2} \left( \sum_{k=1}^{k_n} \left| E \zeta_{nk}^2 + E 6_{nk}^2 \right| / q^s R_{ns}^{s/(1+s)} \right) \leq \]

\[ \leq K_{s,q,f} R_{ns}^{1/(1+s)}, \quad 1 < s \leq 2. \]

Thus from (23) and (24) we get (18).
Lemma 3. If $1 < s \leq 2$, then

$$P(\mathbf{\Phi}_3 > q(\log R_{ns})^{-c/2} R_{ns}^{1/(1+s)}) \leq K_{s,q} R_{ns}^{1/(1+s)}.$$  \hfill (25)

**Proof.** It follows from (15) that for every $x > 0$

$$P(\mathbf{\Phi}_3 > x) \leq P(\Omega_\nu (\log R_{ns})^{-c/2} \sum_{k=1}^{k_n} \int_{T_n,k+1} T_n,k+1 |s - B_{nk}| ds > x).$$

On the other hand

$$\int_{T_n,k+1} T_n,k+1 |s - B_{nk}| ds \leq \int_{T_n,k+1} T_n,k+1 (s - T_{nk}) ds +$$

$$+ \int_{T_n,k+1} T_n,k+1 |T_{nk} - B_{nk}| ds = \tau_{n,k+1}^2 + |T_{nk} - B_{nk}| \tau_{n,k+1}^2.$$

Hence

$$P(\mathbf{\Phi}_3 > x) \leq P(\sum_{k=1}^{k_n} \tau_{nk}^2 > x/\Omega_\nu (\log R_{ns})^{-c/2} \ ] )$$

$$+ P(2 \max_{1 \leq k \leq n} |T_{nk} - B_{nk}| \tau_{nk} \geq x/\Omega_\nu (\log R_{ns})^{-c/2} \ ).$$

But for every $x > 0$

$$P(\max_{1 \leq k \leq k_n} |T_{nk} - B_{nk}| \tau_{nk} > x) \leq P( T_{nk} > x ) +$$

$$+ P(\max_{1 \leq k \leq k_n} |T_{nk} - B_{nk}| > x/2 )$$

and

$$P(\sum_{k=1}^{k_n} \tau_{nk}^2 > x) \leq P( T_{nk} > x \ , \ T_{nk} \leq 2 ).$$
Furthermore

\[ P\left( \sum_{k=1}^{k_n} I(\mathcal{C}_{nk} > x, T_{nk_n} \leq 2) \right) = P\left( \sum_{k=1}^{k_n} I(\mathcal{C}_{nk} > x/4) \right) + \]

\[ + \sum_{k=1}^{k_n} I(\mathcal{C}_{nk} \leq x/4) \]

\[ \leq P\left( \sum_{k=1}^{k_n} I(\mathcal{C}_{nk} > x/4) \right) \]

\[ \leq P\left( \sum_{k=1}^{k_n} I(\mathcal{C}_{nk} > x/4) \right) > x/2, T_{nk_n} \leq 2 \) \] = \( I_n(x) \)

since

\[ P\left( \sum_{k=1}^{k_n} I(\mathcal{C}_{nk} \leq x/4) \right) > x/2, T_{nk_n} \leq 2 \) = 0.

On the other hand

\[ I_n(x) \leq P\left( \sum_{k=1}^{k_n} I(\mathcal{C}_{nk} > x/4) \right) = \]

\[ = P\left( \bigcup_{k=1}^{k_n} I(\mathcal{C}_{nk} > x/4) \right) \leq \sum_{k=1}^{k_n} P(\mathcal{C}_{nk} > x/4) \leq 4s I_s T_n^{(s)} x^{-s}. \]

Thus taking into account the inequalities given above and (17) we get (25).

**Lemma 4.** If \( 1 \leq s \leq 5/3 \), then

\[ P(\mathcal{D} > q(\log R_n)^{-1/2} R_n^{1/(1+s)} ) \leq K_{s,q} R_n^{1/(1+s)}. \]

**Proof.** Taking into account (15) and the considerations given in [3] (proof of Lemma 5) we have

\[ P(\mathcal{D} > x ) \leq P\left( \sum_{k=0}^{k_n-1} \int_{T_{nk}} T_{nk, k+1} |W(s) - W(T_{nk})| ds > x/\sqrt{\log R_n} \right) \]

\[ = P\left( \sum_{k=1}^{k_n} \int_0^1 |W_k(s)| ds > x/\sqrt{\log R_n} \right) \]

\[ > x/\sqrt{\log R_n} \]
\[
\Pr\left( \sum_{k=1}^{k_n} \zeta_{nk} \sup_{s \leq \zeta_{nk}} |\bar{W}_k(s)| > x/2 \right) \leq \Pr\left( \sum_{k=1}^{k_n} \zeta_{nk} \sup_{s \leq \zeta_{nk}} |\bar{W}_k(s)| > x/2 \right),
\]

where \( \bar{W}_k(s) = \bar{W}(s + T_{nk}) - \bar{W}(T_{nk}) \).

On the other hand,
\[
\sum_{k=1}^{k_n} \zeta_{nk} \sup_{s \leq \zeta_{nk}} |\bar{W}_k(s)| = \sum_{k=1}^{k_n} \zeta_{nk} \left( \sup_{s \leq \zeta_{nk}} |\bar{W}_k(s)| - \bar{W}_k(\zeta_{nk}) \right)
\]
\[
+ |\bar{W}_k(\zeta_{nk})| \leq \sum_{k=1}^{k_n} \zeta_{nk} \sup_{s \leq \zeta_{nk}} |\bar{W}_k(s) - \bar{W}_k(\zeta_{nk})|/\zeta_{nk} + \sum_{k=1}^{k_n} \zeta_{nk} |\bar{W}_k(\zeta_{nk})|.
\]

Let us put
\[
U_{nk} = \sup_{s \leq \zeta_{nk}} |\bar{W}_k(s) - \bar{W}_k(\zeta_{nk})|/\zeta_{nk}, \quad 1 \leq k \leq k_n.
\]

Hence, by (27) and (28), we get
\[
\Pr\left( \bar{\Phi}_4 > q(\log R_{ns}^{-1})^{1/2} R_{ns}^{1/(1+s)} \right) \leq \Pr\left( \sum_{k=1}^{k_n} \zeta_{nk} U_{nk} > qR_{ns}^{1/(1+s)}/2 \right) + \Pr\left( \sum_{k=1}^{k_n} |\bar{W}_k(\zeta_{nk})| > qR_{ns}^{1/(1+s)}/2 \right).
\]

Let us observe that for every \( p \geq 2/3 \)
\[
\mathbb{E}(\zeta_{nk} |\bar{W}_k(\zeta_{nk})|)^p \leq (\mathbb{E}(\zeta_{nk})^{3p/2})^{2/3} (\mathbb{E}|\bar{W}_k(\zeta_{nk})|^{3p})^{1/3}
\]
\[
L_{3p/2}^{2/3} (\mathbb{E}|X_{nk}|^{3p})^{2/3} (\mathbb{E}|X_{nk}|^{3p})^{1/3} = L_{3p/2}^{2/3} L_{3p/2}^{3p/3}, \quad 1 \leq k \leq k_n.
\]

Moreover, by Lemma 4 \([3]\), for every \( p \geq 2/3 \)
\[
\mathbb{E}(\zeta_{nk} U_{nk})^p \leq 9_p L_p^{3p-1} \mathbb{E}|X_{nk}|^{3p/(3p-1)}.
\]

Hence, in the case \( 1 < s \leq 3/2 \), we get
(32) \[ P \left( \sum_{k=1}^{kn} \zeta_{nk} U_{nk} > qR_{ns}^{1/(1+s)/2} \Omega \right) \leq (2 \Omega R_{ns}^{2s/3})^{2s/3} \Omega_{ns}^{2s/3} (1+s) \]

\[ \cdot \left( \sum_{k=1}^{kn} E(\zeta_{nk} U_{nk})^{2s/3}/q^{2s/3} \right) \leq (2 \Omega R_{ns}^{2s/3})^{2s/3} \Omega_{ns}^{2s/3} (1+s) \sum_{k=1}^{kn} E(\zeta_{nk} U_{nk})^{2s/3} \]

\[ / q^{2s/3} \leq K_{s,q,s}^{(3+s)/3(1+s)} \leq K_{s,q,s}^{1/(1+s)} \]

and by the same way we prove that

(33) \[ P \left( \sum_{k=1}^{kn} \zeta_{nk} | W_k(\zeta_{nk}) | > qR_{ns}^{1/(1+s)/2} \Omega \right) \leq K_{s,q,s}^{1/(1+s)} \]

Thus, in the case \( 1 < s \leq 3/2 \), (26) follows from (29), (32) and (33).

If \( 3/2 < s \leq 2 \), then taking into account (30), (31) and Lemma 2 [9, p. 139] , which can be proved for our random variables too, we obtain

\[ \left( \sum_{k=1}^{n} E(\zeta_{nk} U_{nk}) \right) \leq C_1 L_n^{3/2} \leq C_1 (L_n(s) - 1/2(s-1)) \]

and

\[ \left( \sum_{k=1}^{n} E(\zeta_{nk} | W_k(\zeta_{nk}) | ) \right) \leq C_2 L_n^{3/2} \leq C_2 (L_n(s) - 1/2(s-1)) \]

where \( C_1 \) and \( C_2 \) are constants depending only on \( s \).

On the other hand

\[ P \left( \sum_{k=1}^{kn} \zeta_{nk} | W_k(\zeta_{nk}) | > qR_{ns}^{1/(1+s)/2} \Omega \right) = P \left( \sum_{k=1}^{kn} \zeta_{nk} | W_k(\zeta_{nk}) | - \right) \]

\[ - \sum_{k=1}^{kn} E(\zeta_{nk} | W_k(\zeta_{nk}) | | G_{n,k-1}) > qR_{ns}^{1/(1+s)/2} \Omega \]

\[ - \sum_{k=1}^{kn} E(\zeta_{nk} | W_k(\zeta_{nk}) | | G_{n,k-1}) > \right) \]

Let \( \zeta_{nk} | W(\zeta_{nk}) | - E(\zeta_{nk} | W_k(\zeta_{nk}) | | G_{n,k-1}) = A_{nk} \)
If $5/2 - s \leq \frac{3}{2}$, then by the definition of random variables $\zeta_{nk}$, $1 \leq k \leq k_n$, and Lemma 1

$$P(A) = P\left(\sum_{k=1}^{k_n} E(\zeta_{nk} | W_k(\zeta_{nk}) | G_{n,k-1}) > q_{ns}^{1/(1+s)} / 4 \Omega_0 \right) \leq$$

$$\leq P\left(\sum_{k=1}^{k_n} [E(\zeta_{nk}^3 | G_{n,k-1})]^{2/3} [E|X_{nk}|^3 | G_{n,k-1}]^{1/3} \right) > q_{ns}^{1/(1+s)} / 4 \Omega_0 \leq P\left(\sum_{k=1}^{k_n} E(|X_{nk}|^3 | F_{n,k-1}) \right)^{1/3} > q_{ns}^{1/(1+s)} / 4 \Omega_0 \leq P\left(\sum_{k=1}^{k_n} E(|X_{nk}|^3 | F_{n,k-1}) \right) \leq K_{s,f} \frac{R_n}{R_{ns}}^{1/(1+s)} \leq K_{s,f} \frac{R_n}{R_{ns}} \leq K_{s,f} \frac{R_n}{R_{ns}}, \quad 3/2 < s \leq 5/3.$$ 

Furthermore, taking into account the inequality

$$E(U_{nk}^3 | G_{n,k-1}) \leq CE(\zeta_{nk} | G_{n,k-1}) \quad \text{a.s.,}$$

where $C$ is an absolute constant, step by step as in above we prove that for sufficiently large $n$

$$P\left(\sum_{k=1}^{k_n} E(\zeta_{nk} U_{nk} | G_{n,k-1}) > q_{ns}^{1/(1+s)} / 4 \Omega_0 \right) \leq K_{s,f} \frac{R_n}{R_{ns}}^{1/(1+s)},$$

for every $1 < s \leq 5/3$. The inequality (34) easily follows from the proof of Lemma 4 [3] and the properties of the random variables $\zeta_{nk}$ and $U_{nk}$.

On the other hand $\{W_{nk}, G_{nk}, 1 \leq k \leq k_n\}$ is a martingale. Thus, by Burkholder's martingale inequality (Theorem 9 [5] or Lemma [6]) there exists a finite universal constant $M_s$ such that
\[
P( w_{nk} > q R_{ns}^{1/(1+s)} / 2 \Omega_0 ) \leq \sum_{k=1}^{k_n} \mathbb{E}( \tau_{nk} U_{nk} \mid G_{n,k-1} ) \leq \mathbb{E}( \tau_{nk} U_{nk} \mid G_{n,k-1} ) \leq q R_{ns}^{1/(1+s)} / 2 \Omega_0
\]

Similarly we prove that if \( 3/2 \leq s \leq 5/3 \), then

\[
P( \sum_{k=1}^{k_n} \mathbb{E}( \tau_{nk} U_{nk} \mid G_{n,k-1} ) ) > q R_{ns}^{1/(1+s)} / 2 \Omega_0 -
\]

Thus the proof of Lemma 4 is ended.

Now (5) follows from (6), (7), (8), (9), (16), (17) and Lemmas 2 - 4 provided \( 1 \leq s \leq 5/3 \). On the other hand if \( s > 5/3 \), then

\[
(35) \quad P( |F_1 - \Phi(W)| \geq 2 q/2 (\log R_{ns}^{-1})^{\ell/2} R_{ns}^{1/4(s-1)} ) \leq P( |F_1 - \Phi(W)| \geq q(\log R_{ns}^{-1})^{\ell/2} (L_{\infty}^s + E[V_n^2 - 1]^3)^{1/4(s-1)} ) \leq P( |F_1 - \Phi(W)| \geq q(\log R_{ns}^{-1})^{\ell/2} ((L_{\infty}^{5/3})^{3(s-1)/2} + (E[V_n^2 - 1]^{5/3})^{3s/5})^{1/4(s-1)} ) \leq P( |F_1 - \Phi(W)| \geq 2(q/2) (\log R_{ns}^{-1})^{\ell/2} (\max \sum (L_{\infty}^{5/3})^{3(s-1)/2} )
\]
Thus if \( s > 5/3 \), then (5) follows from (6) and (35).

4. Proof of Theorem 2. We have

\[
\sup_x |P(\Phi(X_n) \leq x) - P(\Phi(W) \leq x)| \leq \sup_x |P(\Phi(X_n) \leq x) - P(\Phi(W) \leq x)| + \left( p_{\text{min}}^{-1}(1/(1+s), 1/4(s-1)) \right) + \left( E|V_n^2 - 1|^{1/3} \right) + \left( E|V_n^2 - 1|^{1/3} \right).
\]

The estimation of \( I_2 \) gives Theorem 1. On the other hand, by Theorem 1

\[
\sup_x |P(\Phi(W) \leq x) - P(\Phi(X_n) \leq x + \delta)| = K_s,q,f \delta\text{,}
\]

where \( \delta = \delta(s) = (\log R_{ns}^{-1})^p_{\text{min}}(1/(1+s), 1/4(s-1)) \). Moreover it follows from the assumption that the function \( \varphi(x) = P(\Phi(W) \leq x) \) satisfies the Lipschitz condition. Thus from (37)

\[
\sup_x |P(\Phi(W) \leq x + \delta)| \leq C \delta\text{,}
\]

where \( C \) is a constant independent of \( \delta \) and \( n \). Hence, taking into account the proof of Theorem 1 we see that the proof of Theorem 2 will be ended if we show that

\[
P(\left| \Phi(X_n) - F \right| > \delta) \leq C \delta
\]

where \( C \) is an absolute constant.
Let us put
\[ H = | \Phi(X_n) - F |. \]

Then
\[ P(H > \varepsilon) \leq P(H^A > \varepsilon) + P(\max_{1 \leq t \leq 1} |X_n(t)| > A), \]

where \( H^A = HI(\max_{0 \leq t \leq 1} |X_n(t)| \leq A). \) Furthermore, by the definition of the random variables \( T_{nk}, 1 \leq k \leq k_n, \) and (6)
\[ P(\max_{0 \leq t \leq 1} |X_n(t)| > A) = P(\max_{1 \leq k \leq k_n} |W(t)| > A) \leq P(\max_{0 \leq t \leq 2} |W(t)| > A) + P(T_{nk} > 2). \]

Hence and from (17) for \( A = (\log R_n^{-1})^{1/2} \) we get
\[ P(\max_{0 \leq t \leq 1} |X_n(t)| \geq 2(\log R_n^{-1})^{1/2}) \leq K_S \delta(s), \]

where \( K_S \) is a constant depending only on \( s. \)

By the definition of \( H^A \) and (15) we have
\[ H^A \leq (\log R_n^{-1})^{\zeta/2} \sum_{k=0}^{k_n-1} \int_{B_n^k} (t - B_n^{k+1}) dt + \]
\[ + \int_{B_n^{k+1}} |X_n(t) - S_{nk}| dt = \Omega(\log R_n^{-1})^{\zeta/2} \sum_{k=1}^{k_n} b_{nk}^4 + \]
\[ + |X_n| b_{nk}^{\zeta/2}. \]

Hence
\[ P(H^A > \varepsilon) \leq P(\sum_{k=1}^{k_n} |X_n| b_{nk}^2 > 2 \varepsilon (\log R_n^{-1})^{-\zeta/2} / \Omega_n - \]
\[ - \sum_{k=1}^{k_n} b_{nk}^4) = P(\sum_{k=1}^{k_n} (|X_n| - \mathbb{E}(|X_n|) b_{nk}^2) > \]
\[ > 2 \varepsilon (\log R_n^{-1})^{-\zeta/2} / \Omega_n - \sum_{k=1}^{k_n} \mathbb{E}(|X_n|) b_{nk}^2 + b_{nk}^4). \]
On the other hand if $1 < s \leq \frac{5}{3}$, then

$$
(45) \quad P(\sum_{k=1}^{k_n} b_{nk}^4 > q R_{n_s}^{1/(1+s)}) \leq P(\sum_{k=1}^{k_n} b_{nk}^4 > q R_{n_s}^{1/(1+s)}, A),
$$

where $A = \left\{ \sum_{k=1}^{k_n} E(\lvert X_{nk} \rvert^{2s} | P_{n,k-1}) > 1 \right\}$. But $P(A) \leq R_{n_s}$ and

$$
(46) \quad P(\sum_{k=1}^{k_n} b_{nk}^4 > q R_{n_s}^{1/(1+s)}, \overline{A}, \lvert v_n^2 - 1 \rvert \leq 1/2) + P(\lvert v_n^2 - 1 \rvert > 1/2) \leq
$$

$$
\leq P(\sum_{k=1}^{k_n} \{ E(\lvert X_{nk} \rvert^{2s} | P_{n,k-1}) \}^{1/s} > q R_{n_s}^{1/(1+s)/4}, \overline{A} ) + 2^{s_E} \lvert v_n^2 - 1 \rvert^s
$$

$$
\leq P(\sum_{k=1}^{k_n} \{ E(\lvert X_{nk} \rvert^{2s} | P_{n,k-1}) \}^{1/s} > q R_{n_s}^{1/(1+s)} ) + 2^{s_E} \lvert v_n^2 - 1 \rvert^s \leq
$$

$$
\leq 4 q^{-1} R_{n_s}^{s/(s+1)} + 2^{s_E} \lvert v_n^2 - 1 \rvert^s.
$$

Furthermore, if $\frac{3}{2} \leq s \leq \frac{5}{3}$, then

$$
(47) \quad P(\sum_{k=1}^{k_n} E(\lvert X_{nk} \rvert^2 | P_{n,k-1}) b_{nk}^2 > q R_{n_s}^{1/(1+s)}) \leq 2^{s_E} \lvert v_n^2 - 1 \rvert^s +
$$

$$
+ P(\sum_{k=1}^{k_n} E(\lvert X_{nk} \rvert^2 | P_{n,k-1}) b_{nk}^2 > q R_{n_s}^{1/(1+s)}) \leq 2^{s_E} \lvert v_n^2 - 1 \rvert^s +
$$

$$
+ P(\sum_{k=1}^{k_n} \{ E(\lvert X_{nk} \rvert^3 | P_{n,k-1}) \}^{3/2} \sum_{k=1}^{k_n} \{ E(\lvert X_{nk} \rvert^2 | P_{n,k-1}) \}^{3/2} )^{1/3} \sum_{k=1}^{k_n} \{ E(\lvert X_{nk} \rvert^3 | P_{n,k-1}) \}^{3/2} J^{1/3} >
$$

$$
> q R_{n_s}^{1/(1+s)/2} \leq 2^{s_E} \lvert v_n^2 - 1 \rvert^s + P(\sum_{k=1}^{k_n} E(\lvert X_{nk} \rvert^3 | P_{n,k-1}) >
$$

$$
\geq 2^{-1} q R_{n_s}^{1/(1+s)} = 2^{s_E} \lvert v_n^2 - 1 \rvert^s
$$

for sufficiently large $n$. On the other hand, if $1 < s \leq \frac{3}{2}$, then

by (45) and (46)

$$
(48) \quad P( H^A > \delta ) \leq P(\sum_{k=1}^{k_n} b_{nk}^4 > q R_{n_s}^{1/(1+s)/4}) + P(\sum_{k=1}^{k_n} \lvert X_{nk} \rvert E(\lvert X_{nk} \rvert^2 |
$$

Furthermore, if \( \frac{3}{2} \leq s \leq \frac{5}{3} \), then

\[
(49) \quad P(H^k > \delta) \leq P\left( \sum_{k=1}^{n} \left( |X_{nk}| - E(|X_{nk}| |F_{n,k-1}) \right)^2 \right) \geq \frac{1}{q_1} \frac{1}{(1+s)}
\]

\[
- \sum_{k=1}^{n} \left[ E\left(|X_{nk}| |F_{n,k-1})\right)^2 + E\left(|X_{nk}| |F_{n,k-1})\right)^2 \right] \leq K_s, q \frac{1}{(1+s)} + \sum_{k=1}^{n} H_{nk} > q_1 \frac{1}{(1+s)}
\]

where \( H_{nk} = |X_{nk}| - E(|X_{nk}| |F_{n,k-1}) | \frac{2}{s} \) \( i \leq k \leq n \).

One can easily note that \( \{ (\sum_{k=1}^{n} H_{nk} | F_{ni} \}, \ 1 \leq i \leq k, \ n \geq 1, \) is a triangular martingale array. Thus by Burkholder’s martingale inequality \((5)\) or \((7)\)

\[
(50) \quad P\left( \sum_{k=1}^{n} H_{nk} > q_1 \frac{1}{(1+s)} \right) \leq q_1^{-2s/3} R_{ns}^{-2s/3} (1+s) E\left( \sum_{k=1}^{n} H_{nk}^2 \right)^{2s/6} \leq q_1^{-2s/3} M_{2s/3} R_{ns}^{-2s/3} (1+s) E\left( \sum_{k=1}^{n} H_{nk}^2 \right)^{2s/6} \leq q_1^{-2s/3} M_{2s/3} R_{ns}^{-2s/3} (1+s) E\left( \sum_{k=1}^{n} H_{nk}^2 \right)^{2s/6} \leq K_s, q \frac{1}{(1+s)},
\]

for every \( \frac{3}{2} \leq s \leq \frac{5}{3} \). Hence, in the case \( 1 < s \leq \frac{5}{3} \), Theorem 2 follows from \((36)\) - \((50)\) and Theorem 1. If \( s > \frac{5}{3} \), then the proof of Theorem 2 is the same as Theorem 1.

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Introduction

In this work we present a completely general technique for obtaining a solution of the moment problem. By this we mean, given a complex-valued integrand \( h(t, \lambda) \), find necessary and sufficient conditions on a function \( T(t) \) in order that

\[
T(t) = \int_{-\infty}^{\infty} h(t, \lambda) dE(\lambda)
\]

for some non-negative bounded measure \( E(\lambda) \).

A virtue of our reproducing kernel approach is that the method extends with only slight modification to the Hilbert space setting. Thus \( T(t) \) may be an operator family and \( E(\lambda) \) a generalized spectral measure.

Definition. A \(*\)-parameter set \( S \) is a set \( S = \{r, s, t, \ldots\} \) along with an idempotent unary operation \( * \), \( r^{**} = r \), and at least one \(*\)-fixed element \( u, u^* = u \).

Remark. Since any set may be endowed with the required operation \( * \) by setting \( r^* = r \) for all \( r \in S \), it is seen that \( S \) may be completely arbitrary. Nevertheless, in all our applications, \( S \) will have either algebraic or topologic structure.

Definitions. A reproducing kernel space \((H, K)\) is an inner-product space of functions \( \phi(\cdot) : S \to H \) from a \(*\)-parameter set \( S \) into a complex Hilbert space \( H \) along with a kernel \( K(\cdot, \cdot) : S \times S \to L(H) \) defined on pairs of parameters into the linear (possibly unbounded) operators defined on \( H \). Further \( K(\cdot, r)x \in H \) for \( r \in S \) and \( x \in H \) and the reproducing property holds, i.e.,

\[
(\phi(\cdot), K(\cdot, r)x)_H = <\phi(r), x>_H, \quad \phi \in H, \ x \in H, \ r \in S.
\]

In the event that \( H \) is taken as the space of complex numbers, \( H = \mathbb{C} \), the classical Aronszajn case, then \( K(r, s) \) is just a complex number for
each \( r, s \in S \).

Letting \( K^*(r, s) \) denote the adjoint of \( K(r, s) \) when it exists, we say \( K \) is \textbf{Hermitian symmetric} if

\[
K^*(r, s) = K(s, r), \quad r, s \in S.
\]

We say \( K \) \textbf{factors} if, for the \( * \)-fixed element \( u \),

\[
K(r, s) = K(r, u)K(u, s), \quad r, s \in S.
\]

Finally, \( K \) is \textbf{positive definite} if

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \langle K(r_i, r_j)x_j, x_i \rangle \geq 0
\]

for all \( n = 1, 2, \ldots \), \( \{r_1, \ldots, r_n\} \subset S \), and \( \{x_1, \ldots, x_n\} \subset H \).

**Definitions.** An operator \( \tilde{T} \) in a Hilbert space \( \tilde{H} \) is a \textbf{dilation} of the operator \( T \) in the Hilbert space \( H \), notation \( T = \text{pr}\tilde{T} \), if \( \tilde{H} \supset H \) and for \( x \in \text{dom} \ T \),

\[
\tilde{P}\tilde{T}x = Tx
\]

where \( \tilde{P} \) is the orthogonal projection of \( \tilde{H} \) onto \( H \).

**Remark.** If \( T = \text{pr}\tilde{T} \), then on \( H \), \( T \), and \( \tilde{T} \) have the same weak values, i.e.,

\[
\langle Tx, y \rangle = \langle \tilde{T}x, y \rangle, \quad x \in \text{dom} \ T, \ y \in H.
\]

**Definitions.** Let \( T \in \mathcal{L}(H)^S \) be a collection of functions \( T(\cdot): S \to \mathcal{L}(H) \). A function \( F(\cdot, \cdot, \cdot): T \times S^2 \to \mathcal{L}(H) \) is a \textbf{linear spread} on \( T \) if \( F \) has the form

\[
F(T(\cdot), r, s) = \sum_{p=1}^{N} c_p(r, s)T(g_p(r, s)), \quad r, s \in S
\]

for some complex valued functions \( c_p: S^2 \to \mathbb{C} \) and binary operations \( g_p: S^2 \to S \), \( p = 1, \ldots, N \).

In this event define \( F^- \) to be

\[
F^-(T(\cdot), r, s) = \sum_{p=1}^{N} \overline{c_p(r, s)}T(g_p(r, s)), \quad r, s \in S.
\]

**Definition.** A linear spread \( F \) is \textbf{admissible for} \( T(\cdot): S \to \mathcal{L}(H) \) if the following equalities are satisfied, for \( r, s, t, w \in S \),

\begin{align*}
(\alpha 1) \quad & F(T(\cdot), r, u) = T(r), \\
(\beta 1) \quad & F(F(T(\cdot), \cdot, t), r, r^*, s*) = F(F(T(\cdot), \cdot, t)s, r), \\
(\beta 2) \quad & F(F(V(\cdot), \cdot, t), r, s) = F(F(V(\cdot), r, \cdot), s, t),
\end{align*}

where \( V(\cdot) = F(T(\cdot), w, \cdot) \),

and

\begin{align*}
(\alpha 2) \quad & F(V(\cdot), u, s) = V(s), \quad \text{where} \ V(\cdot) = T(\cdot) \quad \text{or} \quad V(\cdot) = F(T(\cdot), r, \cdot),
\end{align*}
or $V(\cdot) = F(F(T(\cdot),r,s),r,s)$.  

Remark. (due to P. Masani). One interpretation of a linear spread is an "action" of the set $S \times S$ on $L(H)$. Surpressing $F$ by means of the notation $(r,s) \otimes T(\cdot)$ to mean $F(T(\cdot),r,s)$, we then may write three of the admissibility conditions as follows.

\[(a1), (a2) \]
\[(u,r) \otimes T = (r,u) \otimes T = \varepsilon_r \text{ (evaluation of } T \text{ at } r), \]
and \[(b2) \ (r,s) \otimes ((\cdot,t) \otimes T) = (s,t) \otimes ((r,\cdot) \otimes T), \]

nevertheless, with a view toward our intended application, we prefer the original notation.

Examples

1) Let $S$ be the set $S = \{u,r,s,t\}$ with binary operation $\cdot$ given by Table 1, and $v^* = v$ for all $v \in S$. Define the linear spread $F$ on the class of all functions $T : S \rightarrow L(H)$ by $F(T(\cdot),v,w) = T(v \cdot w)$.

\[
\begin{array}{cccc}
\cdot & u & r & s & t \\
\hline
u & u & r & s & t \\
r & r & u & r & s \\
s & s & r & u & t \\
t & t & s & t & u \\
\end{array}
\]

\text{TABLE 1.}

The calculations

$F(F(T(\cdot),\cdot,s),r,t) = F(T(\cdot),r \cdot t,s) = T((r \cdot t) \cdot s) = T(u)$

and

$F(F(T(\cdot),r,\cdot),t,s) = F(T(\cdot),r \cdot t,s) = T(r \cdot (t \cdot s)) = T(s)$

show that in general $(b2)$ is not satisfied.

2) Let $S$ be any $*$-semi-group, i.e., $S$ has an associative binary operation $\cdot$, an identity $u$, and an idempotent unary operation $*$ such that $(r \cdot s)^* = s^* \cdot r^*$ and $u^* = u$. Define the linear spread $F$ just as in 1) above. Simple calculations show that $F$ is now admissible. In fact for $b1$ we have

$F(F(T(\cdot),\cdot,t),r^*,s^*) = F(T(\cdot),(r^* \cdot s^*)^*,t) = T((s \cdot r) \cdot t)$

while
3) Let $S = \mathbb{R}$, the real numbers, let $r^* = -r$ and $u = 0$. Let $T$ be the class of all functions $T: S \to L(H)$ such that $T(-r) = T(r)$. Define $F$ by

$$F(T(\cdot), r, s) = \frac{1}{2} T(s+r) + \frac{1}{2} T(s+r^*).$$

Again straightforward calculations show $F$ is admissible on $T$.

**Theorem.** Let $F$ be a linear spread admissible for $T$, $T: S \to L(H)$, and define the kernel $K(\cdot, \cdot): S \times S \to L(H)$ by

$$K(r, s) = F(T(\cdot), r^*, s).$$

If $T(u) = I$ and $K$ is positive definite, then $T$ has a dilation $\tilde{T}$ for which $F$ is admissible. Further, the kernel

$$\tilde{K}(r, s) = F(\tilde{T}(\cdot), r^*, s)$$

is Hermitian symmetric and factors,

$$\tilde{K}(r, s) = \tilde{T}^*(r)\tilde{T}(s) = \tilde{T}(r^*)\tilde{T}(s).$$

**Proof.** Following the usual Moore construction, let $H$ be the completion of the inner-product space $H_0$ of functions $\psi(\cdot): S \to H$ of the form

$$\psi(\cdot) = \sum_{i=1}^{n} K(\cdot, r_i)x_i, \quad r_i \in S, \ x_i \in H.$$

We may embed $H$ in $H_0$ by identifying

$$x \mapsto K(\cdot, u)x.$$

Next define $\tilde{T}(s)$ formally on $H_0$ by setting

$$\tilde{T}(s) = \sum_{i=1}^{n} K(t, w_i)x_i = \sum_{i=1}^{n} F(K(t, \cdot), s, w_i)x_i = \sum_{i=1}^{n} \sum_{p=1}^{N} c_p(x, w_i)K(t, q_p(s, w_i))x_i. \quad (1)$$

Also define $\tilde{K}(r, s)$ formally on $H_0$ by setting

$$\tilde{K}(r, s) = \sum_{i=1}^{n} K(t, w_i)x_i = F(\tilde{T}(\cdot), r^*, s) = \sum_{i=1}^{n} K(t, w_i)x_i = \sum_{p=1}^{N} c_p(r^*, s)\tilde{T}(q_p(r^*, s)) = \sum_{i=1}^{n} K(t, w_i)x_i. \quad (2)$$

From (1) we see that $\tilde{T}(s)\psi(\cdot) \in H_0$ when $\psi \in H_0$, and from (2) similarly...
\( \tilde{K}(r,s)\phi(\cdot) \in H_0 \) when \( \phi \in H_0 \). At this time we must regard these definitions as formal definitions until it can be shown that they are independent of the particular representations of the functions \( \phi \in H_0 \).

First we show that formally, \( \tilde{T}(s) = \tilde{K}(u,s) \) on \( H_0 \). For this it is sufficient to show that \( \tilde{K}(u,s)K(t,w)x = \tilde{T}(s)K(t,w)x \) for all \( w \in S \) and \( x \in H \). Starting with the definition (2) and using linearity of \( F \) in its first argument along with condition \( \beta 2 \) we have,

\[
\tilde{K}(u,s)K(t,w)x = F(\tilde{T}(\cdot),u*,s)K(t,w)x \\
= F(\tilde{T}(\cdot)K(t,w),u,s)x = F(F(K(t,\cdot),*,w)u,s)x \\
= F(F(K(t,\cdot),u,\cdot),s,w)x.
\]

Now using condition \( \alpha 2 \) on this latter expression yields the required result,

\[ F(K(t,\cdot),s,w)x = \tilde{T}(s)K(t,w)x. \]

Also, it is valid that \( \tilde{K}(r,u) = \tilde{K}(r*) \) as we show in a manner similar to the argument above.

\[
\tilde{K}(r,u)K(t,w)x = F(\tilde{T}(\cdot),r*,u)K(t,w)x \\
= F(\tilde{T}(\cdot)K(t,w),r*,u)x = F(F(K(t,\cdot),*,r,u)x \\
= F(F(K(t,\cdot),r,\cdot),u,w)x.
\]

Again using condition \( \alpha 2 \) with the function \( F(K(t,\cdot),r*,\cdot) \) we derive the required result,

\[ F(K(t,\cdot),r*,w)x = \tilde{T}(r*)K(t,w)x. \]

Next we prove that formally, for \( \phi, \psi \in H_0 \),

\[
(\tilde{K}(s,u)\phi(\cdot),\psi(\cdot)) = (\phi(\cdot),\tilde{K}(u,s)\psi(\cdot)).
\]

Without loss of generality we may assume \( \phi(\cdot) = K(\cdot,r)x \) and \( \psi(\cdot) = K(\cdot,w)y \). Consider the right hand side.

\[
(K(t,r)x,\tilde{K}(u,s)K(t,w)y) = (K(t,r)x,F(K(t,\cdot),s,w)y) \\
= (K(t,r)x, \sum_{p=1}^{n} c_p(s,w)K(t,g_p(s,w))y) \\
= \sum_{p=1}^{n} c_p(s,w)F(K(g_p(s,w),r)x,y) \\
= <F^{-}(K(\cdot,r),s,w)x,y> \\
= <F^{-}(F(T(\cdot),\cdot*,r),s,w)x,y>.
\]

By using \( \beta 1 \) and then \( \beta 2 \) the latter becomes
This proves the assertion.

To obtain the factorization property formally, consider the

\[ F(F(t(·),w*,·),s*,r)x,y > = F(F(F(t(·),w*,·),s*,r)x,y) \]
\[ = (F(F(t(·),w*,·),s*,r)x,K(t,w)y) \]
\[ = (T(s*)K(t,r)x,K(t,w)y) \]
\[ = (K(s,u)K(t,r)x,K(t,w)y) \]

In this chain we have used the linearity of \( F \) in its first argument.

Now apply condition \( β2 \) to give

\[ F(F(K(t,·),r*,·),s*)x,F(K(t,·),r*,·),s*)x \]
\[ = K(r,s)K(t,w)x F(T(·),r*,s)K(t,w)x F(T(·)K(t,w),r*,s)x \]
\[ = F(F(K(t,·),·,w),r*,s)x. \]

In this chain we have used the linearity of \( F \) in its first argument.

Factorization on \( H_0 \) now follows by extending this calculation to sums

\[ \phi(·) = \sum_{i=1}^{n} K(t,r_i)x_i. \]

As a consequence of what we have proved so far, we obtain also

Hermitian symmetry formally. In fact, for \( \phi \in H_0 \), also \( K(r,s)\phi \),
\( K(r,u)\phi \), and \( K(u,s)\phi \in H_0 \) and similarly for \( \psi \in H_0 \). Therefore

\[ (K(r,s)\phi(·),\psi(·)) = (K(r,u)K(u,s)\phi(·),\psi(·)) \]
\[ = (K(u,s)\phi(·),K(u,r)\psi(·)) \]
\[ = (\phi(·),K(s,u)K(u,r)\psi(·)) \]
\[ = (\phi(·),K(s,r)\psi(·)). \]

Finally, we can now show in a simple way that the formal defini-
tions are in fact well-defined. Suppose that

\[ \sum_{i=1}^{n} K(t,r_i)x_i = \sum_{j=1}^{m} K(t,s_j)y_j \]

are two representations for the same function \( \phi(·) \in H_0 \). Then for any
\( \psi(·) \in H_0 \) we have

\[ (\tilde{K}(r,s)(\sum_{i=1}^{n} K(t,r_i)x_i - \sum_{j=1}^{m} K(t,s_j)y_j),\psi(·)) \]
\[ = (\sum_{i=1}^{n} K(t,r_i)x_i - \sum_{j=1}^{m} K(t,s_j)y_j,\tilde{K}(s,r)\psi(·)) = 0 \]

Since \( H_0 \) is dense in \( H \), it follows that
\[ \tilde{K}(r,s) = \sum_{i=1}^{n} K(t,r_i)x_i = \tilde{K}(r,s) = \sum_{j=1}^{m} K(t,s_j)y_j. \]

To see that \( \tilde{T}(r) \) is a dilation of \( T(r) \), \( r \in S \), it suffices to show that for \( x, y \in H \),
\[ (\tilde{T}(r)K(t,u)x, K(t,u)y) = (K(t,u)T(r)x, K(t,u)y). \]

Starting with the left hand side and using first \( \alpha_2 \) and then \( \alpha_1 \) applied to \( T(\cdot) \) we obtain
\[ (F(K(t,\cdot),r,u)x, K(t,u)y) = <F(K(u,\cdot),r,u)x,y> \]
\[ = <F(F(T(\cdot),u*,\cdot),r,u)x,y> \]
\[ = <F(T(\cdot),r,u)x,y> \]
\[ = <T(r)x,y>. \]

As this is also the right hand side, the assertion is proved.

Finally, it remains to see that \( F \) is admissible on \( \tilde{T} \). We have already shown both that \( F(\tilde{T}(\cdot),u,s) = \tilde{T}(s) \), as this is just equation (2) with \( r = u \), and that \( F(\tilde{T}(\cdot),r,u) = \tilde{K}(r*,u) = \tilde{T}(r) \). We finish by proving condition \( \beta_1 \) for \( T(\cdot) \); the proof of condition \( \beta_2 \) is similar and is left for the reader. Thus, for the left side we have
\[ F^{-1}(F(\tilde{T}(\cdot),\cdot*,w),r*,s*) = F^{-1}(\tilde{K}(\cdot,\cdot)*\tilde{T}(w)) = F^{-1}(\tilde{K}(\cdot,\cdot)*\tilde{T}(w)) = F^{-1}(\tilde{K}(\cdot,\cdot)*\tilde{T}(w)) = F^{-1}(\tilde{K}(\cdot,\cdot)*\tilde{T}(w)) = F^{-1}(\tilde{K}(\cdot,\cdot)*\tilde{T}(w)) = F^{-1}(\tilde{K}(\cdot,\cdot)*\tilde{T}(w)). \]

Analogously, the right hand side can be shown to be the same.

**Definition.** A complex-valued function \( h(\cdot,\cdot) : S \times \mathbb{R} \to \mathbb{C} \) is a product functional solution for the linear spread \( F \) if
\[ F(h(\cdot,\cdot),r,s) = h(r,\lambda)h(s,\lambda) \]
and
\[ F(\tilde{h}(\cdot,\cdot),r*,s*) = \tilde{h}(s,\lambda)\tilde{h}(r,\lambda). \]
Examples

1) Consider the $\ast$-semi-group $S = \{0, 1, 2, \ldots\}$ under $+$ with $r^{\ast} = r$ and $u = 0$. Then $h(r, \lambda) = \lambda^r$ is a product functional solution for $F(f(\cdot), r, s) = f(r+s)$. In fact

$$F(h(\lambda, r), r, s) = h(r+s, \lambda) = \lambda^{r+s} = \lambda^r \lambda^s = h(r, \lambda) h(s, \lambda)$$

and

$$F(h(\lambda, r), r^*, s^*) = h((r^*+s^*)^*, \lambda) = \lambda^{r+s} = h(s, \lambda) h(r, \lambda).$$

2) For the $\ast$-semi-group $S = \mathbb{R}$ under $+$ with $r^{\ast} = -r$ and $u = 0$, the function $h(r, \lambda) = e^{ir\lambda}$ is a product functional solution for $F(f(\cdot), r, s) = f(r+s)$. In fact

$$F(h(\lambda, r), r, s) = e^{i(r+s)\lambda} = e^{ir\lambda} e^{is\lambda} = h(r, \lambda) h(s, \lambda)$$

and

$$F(h(\lambda, r), r^*, s^*) = h((r^*+s^*)^*, \lambda) = e^{i(r+s)\lambda} = e^{ir\lambda} e^{is\lambda} = h(s, \lambda) h(r, \lambda).$$

3) On the $\ast$-semi-group $S = \mathbb{R}$ under $r \cdot s = r+s+ars$ (fixed $a \neq 0$) with $r^{\ast} = r$ and $u = 0$, the function $h(r, \lambda) = (1+ar)^\lambda$ is a product functional solution for the spread $F(f(\cdot), r, s) = f(r+s+ars)$. In fact

$$F(h(\lambda, r), r, s) = (1+a(r+s+ars))^{\lambda} = (1+ar+as+a^2rs)^{\lambda} = (1+ar)(1+as)^{\lambda} = h(r, \lambda) h(s, \lambda).$$

Similarly, the second condition also holds.

4) Let $S = (\mathbb{C} - \mathbb{R}) \cup \{0\}$ with $r^{\ast} = \overline{r}$ and $u = 0$. For

$$F(f(\cdot), r, s) = \frac{r}{r-s} f(r) + \frac{s}{s-r} f(s)$$

the function $h(r, \lambda) = \lambda/(1-r\lambda)$ is a product functional solution. Consider the second condition, we leave the first to the reader.

$$F(h(\lambda, r), r^*, s^*) = h(r^*, \lambda) = \lambda^{r^*} = \frac{\lambda^{r^*}}{\lambda^{r^*}} h(s^*) = \frac{\lambda^{r^*}}{\lambda^{r^*}} h(s^*) = \frac{\lambda^{r^*}}{\lambda^{r^*}} h(s^*) = \frac{\lambda^{r^*}}{\lambda^{r^*}} h(s^*)$$

It is not a coincidence that those linear spreads which admit a product functional solution are also admissible.

**Definition.** Given a complex-valued function $T: S \to \mathbb{C}$ and an integer $n > 0$, we say that $h: S \times \mathbb{R} \to \mathbb{C}$ is an $n$-interpolant for $T$ if for any
n values \( t_1, \ldots, t_n \in S \), there exists finitely many values \( \lambda_1, \ldots, \lambda_m \in \mathbb{R} \) and complex numbers \( \beta_1, \ldots, \beta_m \) such that

\[
T(t_i) = \sum_{j=1}^{m} \beta_j h(t_i, \lambda_j), \quad i = 1, \ldots, n.
\]

If \( n \) is understood we simply say \( h \) is an interpolant for \( T \).

**Remark.** We conjecture that a bi-variate function \( h \) will be an interpolant for \( T \) as above if for some \( \lambda, \) 1) \( h(s, \lambda) \neq 0 \) whenever \( T(\lambda) \neq 0 \), and 2) \( h(s, \lambda) \neq h(t, \lambda) \) whenever \( T(s) \neq T(t) \). However, we have been unable to prove this.

**Theorem.** Let \( h: S \times \mathbb{R} \to \mathbb{C} \) be a product functional solution for the linear spread \( F: \mathbb{C}^S \times S^2 \to \mathbb{C} \),

\[
F(f(\cdot),r,s) = \sum_{p=1}^{N} c_p(r,s)f(g_p(r,s)).
\]

Suppose further that \( h(u,t) = 1 \) and that \( h \) is a 3N\(^2\)-interpolant for the complex-valued \( T: S \to \mathbb{C} \). Then \( F \) is admissible for \( T \).

**Proof.** First observe that equalities (a1), (a2), (b1), and (b2) hold for \( h(\cdot, \lambda) \) by direct calculation. Thus, for (b1) consider the left-hand side

\[
F^{-1}(F(h(\cdot, \lambda), *, t), r*, s*) = F^{-1}(h(\cdot, \lambda)h(t, \lambda), r*, s*)
\]

\[
= \left( \sum_{p=1}^{N} c_p(r*, s*) h(g_p(r*, s*), \lambda) \right) h(t, \lambda)
\]

\[
= F(h(\cdot, \lambda), r*, s*)h(t, \lambda)
\]

\[
= h(s, \lambda)h(r, \lambda)h(t, \lambda)
\]

But the right-hand side works out the same,

\[
F(F(h(\cdot, \lambda), \cdot, t), s, r) = F(h(\cdot, \lambda)h(t, \lambda), s, r)
\]

\[
= F(h(\cdot, \lambda), s, r)h(t, \lambda)
\]

\[
= h(s, \lambda)h(r, \lambda)h(t, \lambda).
\]

Next, by linearity of \( F \) in its first argument, the equalities continue to hold for every function \( f: S \to \mathbb{C} \) of the form

\[
f(t) = \sum_{j=1}^{m} \beta_j h(t, \lambda_j), \quad \beta_j \in \mathbb{C}, \lambda_j \in \mathbb{R}.
\]

Now to show that (b1) is satisfied by \( T(\cdot) \), fix \( r, s, t \in S \) and consider the left-hand side of (b1). Written out explicitly it is
Put \( v_{pq} = g_q (r^*, s^*) \), \( p, q = 1, \ldots, N \). Similarly, writing out the right-hand side of (3) above such that \( f \) agrees with \( T \) on the \( 2N^2 \) values \( v_{pq}, w_{pq} \). Therefore from the validity of (3) for \( f \) we have

\[
F^{-\infty} (F(T(\cdot), *, t), r^*, s^*) = F^{-\infty} (F(f(\cdot), *, t), r^*, s^*) = F(f(\cdot), t), s, r) = F(T(\cdot), t), s, r).
\]

This proves that (3) is satisfied by \( T(\cdot) \). In a similar way the other admissibility conditions are also seen to be satisfied by \( T(\cdot) \) and the theorem is proved.

This result provides an alternate way of satisfying the hypothesis of our principle theorem. We write it explicitly as our next result whose proof is immediate.

**Theorem.** Let the complex-valued linear spread \( F, F(f(\cdot), r, s) \), have a product functional solution \( h \) which is a \( 3N^2 \)-interpolant for \( T: S \to \mathbb{C} \). If \( T(u) = 1 \) and the kernel

\[
K(r, s) = F(T(\cdot), r^*, s)
\]

is positive definite, then \( T \) has a dilation \( \tilde{T} \) on which \( F \) is admissible. Further, the kernel \( \tilde{K}(r, s) = F(\tilde{T}(\cdot), r^*, s) \) is Hermitian symmetric and factors.

The relationship between a complex-valued function \( f \), a bivariate function \( h \) an interpolant for it and the linear spread for which \( h \) is a product functional solution is much deeper than the previous theorem portrays. Indeed, the interplay among these functions constitutes a general solution of the moment problem.

**Theorem.** Let \( h(t, \lambda) \) be the unique measurable in \( \lambda \) product functional solution of a linear spread \( F \). Then necessary and sufficient conditions on a complex-valued function \( f: S \to \mathbb{C} \) in order that \( f \) admit a representation
\[ f(t) = \int_{-\infty}^{\infty} h(t, \lambda) dm(\lambda) \]

for some non-negative bounded measure \( m \) are that 1) \( h \) be an interpolant for \( f \) or 1') \( F \) be admissible for \( f \) and 2) the kernel \( K(r,s) = F(f(\cdot), r^*, s) \) be positive definite.

**Proof.** Let \( \tilde{T} \) and \( \tilde{K} \) be as in the principle theorem. By an extension of a result due to A. Devinatz [2], there exists a spectral measure \( \tilde{E}_{\lambda}, -\infty < \lambda < \infty \), and complex-valued functions \( a(r, \lambda) \) such that \( a(u, \lambda) = 1 \) and

\[
\tilde{K}(r,s) = \int_{-\infty}^{\infty} \overline{a(r, \lambda)} a(s, \lambda) d\tilde{E}_{\lambda}, \quad r, s \in S.
\]

We show that \( a(\cdot, \lambda) \) is a product functional solution for \( F \). For each \( \phi, \psi \in H_0 \)

\[
\int_{-\infty}^{\infty} a(r, \lambda) a(s, \lambda) d(\tilde{E}_{\lambda} \phi, \psi) = (\tilde{K}(r, s) \phi, \psi)
\]

\[
= (F(\tilde{K}(u, \cdot), r^*, s) \phi, \psi) = F((\tilde{K}(u, \cdot) \phi, \psi), r^*, s)
\]

\[
= F(\int_{-\infty}^{\infty} a(u, \lambda) a(\cdot, \lambda) d(\tilde{E}_{\lambda} \phi, r^*, s))
\]

\[
= \int_{-\infty}^{\infty} F(a(\cdot, \lambda), r^*, s) d(\tilde{E}_{\lambda} \phi, \psi).
\]

Since this holds for all \( \phi, \psi \in H_0 \), operator

\[
\int_{-\infty}^{\infty} (F(a(\cdot, \lambda), r^*, s) - a(r, \lambda) a(s, \lambda)) d\tilde{E}_{\lambda}
\]

is the zero operator on \( H_0 \). It follows that

\[
F(a(\cdot, \lambda), r, s) = a(r, \lambda) a(s, \lambda).
\]

By uniqueness \( h(r, \lambda) = a(r, \lambda) \) and the conclusion holds with \( m(\lambda) = (\tilde{E}_{\lambda} K(\cdot, u) l, K(\cdot, u) l) \).

**Application.** Consider the integrand \( h(t, \lambda) = e^{\lambda t^2} \) and note that it is the unique product functional solution for the linear spread

\[
F(f(\cdot), r, s) = f(\sqrt{r^2 + s^2}),
\]

where \( r, s \geq 0, r^* = r \), and \( u = 0 \), cf. Aczel [1]. Straightforward calculations show that \( F \) is admissible for any function \( f \). Therefore, by the theorem, the necessary and sufficient condition on \( f \) in order that
it admit a representation

\[ f(t) = \int_{-\infty}^{\infty} e^{\lambda t^2} \, dm(\lambda) \]

for some non-negative bounded measure \( m \) is that the kernel \( K(r,s) = f(\sqrt{r^2 + s^2}) \) be positive definite.

Literature

AN ABSTRACT FORM OF A COUNTEREXAMPLE OF MAREK KANTER.

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1. Introduction.

In his paper [3] M. Kanter gave an example of a probability measure $\mu$ on $\mathbb{R}^T$ such that some linear $\mu$-measurable functionals were not almost sure limits of sequences of continuous linear functionals. In his example $\mu$ is the distribution of a stochastic process $X_t = W_t + Y_t - Z_t$, where $W_t, Y_t, Z_t$ are independent, $W_t$ is a Wiener process and $Y_t, Z_t$ are Poisson processes. The Kanter's example was generalized by K. Urbanik [7], who proved the same for $\mu$ induced by any symmetric, homogeneous, separable and continuous in probability process with independent increments and having both Gaussian and non-Gaussian components. In this note we give an abstract form of this example (Proposition 1). On the other hand Kanter proved in [3] that for a symmetric Gaussian measure every measurable linear functional is an almost sure limit of a sequence of continuous linear functionals. In [7] Urbanik called this a "Riesz property" of a probability measure on an infinite dimensional linear topological space. P. Begziz [1], M. Kanter [4], K. Urbanik [7] proved that a distribution of a process with independent increments, symmetric, homogeneous, separable and continuous in probability and without a Gaussian component has the Riesz property. In this note Proposition 2 and Proposition 3 constitute an abstract form of this fact. In [5] W. Slowikowski introduced "linear Lusin measurable functionals". He proved that they are exactly almost sure limits of sequences of continuous linear functionals. J. Hoffmann-Jørgensen investigated them in case of a product measure [2]. For some results about linear Lusin measurable functionals in case of a cylinder measure see [6].

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2. A general scheme of a measure without the Riesz property.

**Proposition 1.**
Let \( \mu \) and \( \nu \) be symmetric Radon probability measures on a locally convex space \( E \). Suppose that \( \mu \) is not the unit mass at the origin and that the topology of the convergence in \( \nu \) on the topological dual \( E' \) of \( E \) is stronger than that of the convergence in \( \mu \). Then a sufficient condition to ensure that \( \lambda = \mu \ast \nu \) and \( \rho = \frac{1}{2}(\mu + \nu) \) do not have the Riesz property is the existence of linear Borel subspaces \( E_1 \) and \( E_2 \) of \( E \) such that\
\[
\mu(E_1) = \nu(E_2) = 1 \quad \text{and} \quad E_1 \cap E_2 = \{0\}.
\]

**Proof.** Since \( \mu \) and \( \nu \) are tight we can assume that \( E_1 \) and \( E_2 \) are \( \sigma \)-compact. Then \( E_0 = E_1 + E_2 \) is also \( \sigma \)-compact and\
\[
\mu(E_0) = \nu(E_0) = \lambda(E_0) = \rho(E_0) = 1.
\]
Thus with no loss of generality we can assume that \( E_1, E_2 \) are \( \sigma \)-compact and \( E = E_1 + E_2 \). It follows that there exists a linear Borel map \( T : E \to E \) such that \( T(x + y) = x \), for \( x \) from \( E_1 \) and \( y \) from \( E_2 \). Let \( g \) be an element from \( E' \) such that \( \mu(g \neq 0) \neq 0 \). Then \( f = g \circ T \) is a linear Borel functional on \( E \). We claim that \( f \) is not a limit of a sequence of elements of \( E' \) \( \lambda \)-almost surely (\( \lambda \)-a.s.) or \( \rho \)-a.s.

In case of \( \rho \) it is very easy to see it. Indeed, if \( (f_k) \) tends to \( f \) in \( \rho \) then, a fortiori, it tends to \( f \) in \( \nu \). But \( f = 0 \) \( \nu \)-a.s. Thus \( (f_k) \) tends to \( 0 \) in \( \mu \) (assumption). Since \( f = g \mu \)-a.s. we get: \( \mu(g \neq 0) = 0 \). Contradiction. Let now \( X \) and \( Y \) be independent \( E \)-valued random variables defined on a probability space \( (\Omega, \mathcal{M}, P) \) such that \( L(X) = \mu \) and \( L(Y) = \nu \). Suppose that a sequence \( (f_k) \) of elements of \( E' \) converges to \( f \) in \( \lambda \). We have:
\[
f_k(X) + f_k(Y) = f_k(X + Y) + f(X + Y) = g(X),
\]
where equalities hold \( P \)-a.s. and the convergence is in probability \( P \). On the other hand
\[
P(|f_n(X + Y) - f_k(X + Y)| > \epsilon) > \frac{1}{2}P(|(f_n - f_k)(X)| > \epsilon).
\]
It follows that \( (f_k(X)) \) converges to a random variable \( X' \) and \( (f_k(Y)) \) to \( Y' \), where \( X' + Y' = g(X) \) and \( X' \) is \( X \)-measurable. Thus \( Y' \) is \( 0 \) (symmetricity). Hence \( f_k \) converges to \( 0 \) in \( \nu \) which implies that it converges to \( 0 \) in \( \mu \). Thus \( g = 0 \) \( \mu \)-a.s. Contradiction.

**Remark.**
In Kanter's example (cf. Introduction) we have, of course, \( \nu = L(W_t) \) and \( \mu = L(Y_t - Z_t) \). For \( E \) we can take \( L_2(0,1) \) instead of \( R(0,1) \) to get \( \mu \) and \( \nu \) tight.
3. How to build a complicated measure with the Riesz property from a simple one.

Proposition 2.
Let \( \mu \) be a Radon probability measure on a locally convex space \( E \). Then \( \mu \) has the Riesz property iff \( \nu = \text{Poiss} \mu \) has the Riesz property.

**Proof.** Let us recall first that \( \text{Poiss} \mu = e^{-1} \sum_{n=0}^{\infty} \frac{\mu^n}{(n!)^n} \). Let \( f \) be a linear functional on \( E \). Then \( f \) is \( \mu \)-measurable iff there exists a linear Borel subspace \( E_0 \) of \( E \) such that \( \mu(E_0) = 1 \) and \( f \) restricted to \( E_0 \) is Borel. Since \( \mu(E_0) = 1 \) iff \( \nu(E_0) = 1 \) it follows that \( f \) is \( \mu \)-measurable iff it is \( \nu \)-measurable. It is also easy to see that \( (f_k) \) converges to \( f \) in \( \mu \) iff it does so in \( \nu \) and the proposition follows.

Proposition 3.
Let \( \mu \) and \( \nu \) be Radon probability measures on a locally convex space \( E \). Assume that \( \mu \) is symmetric and that for every linear Borel subspace \( E_0 \) of \( E \) if \( \mu(E_0) = 1 \) then \( \nu(E_0) = 1 \).

Under the above assumptions the following conditions are equivalent:

a) \( \mu \) has the Riesz property;

b) \( \lambda = \mu * \nu \) has the Riesz property;

c) \( \rho = \frac{1}{2}(\mu + \nu) \) has the Riesz property.

**Proof.**
a) \( \Rightarrow \) b) and a) \( \Rightarrow \) c):

Let \( f \) be a linear \( \lambda \)-measurable (resp. \( \rho \)-measurable) functional on \( E \). There exists a linear Borel subspace \( E_0 \) of \( E \) such that \( \lambda(E_0) = 1 \) (resp. \( \rho(E_0) = 1 \)) and \( f \) restricted to \( E_0 \) is Borel. We have

\[
1 = \lambda(E_0) = E\mu(E_0 + x)\nu(E_0 - x),
\]

summation being taken over all cosets of \( E_0 \).

(Resp. \( 1 = \rho(E_0) = \frac{1}{2}(\mu(E_0) + \nu(E_0)) \)). Thus \( \mu(E_0) = \nu(E_0) = 1 \). Hence \( f \) is \( \mu \)-measurable and there exist a linear Borel subspace \( E_1 \) of \( E \) and a sequence \( (f_k) \) of elements of \( E' \) such that \( \mu(E_1) = 1 \) and \( (f_k(e)) \) converges to \( f(e) \) for every \( e \) from \( E_1 \). Since \( \nu(E_1) = \mu(E_1) = 1 \) \( (f_k) \) converges to \( f \) \( \lambda \)-a.s. (resp. \( \rho \)-a.s.).

b) \( \Rightarrow \) a) and c) \( \Rightarrow \) a):

Assume that \( \mu \) does not have the Riesz property. Let \( f \) be a linear \( \mu \)-measurable functional on \( E \) which is not an a.s. limit of a sequence of elements of \( E' \). There exists a linear Borel subspace \( E_0 \) of \( E \) such that \( \mu(E_0) = 1 \) and \( f \) restricted to \( E_0 \) is Borel. It follows that \( \lambda(E_0) = \rho(E_0) = 1 \) and that \( f \) is both \( \lambda \)-measurable and \( \rho \)-measurable. If a sequence \( (f_k) \) of
elements of $E'$ converged to $f$ $\lambda$-a.s. or $\rho$-a.s. it would also converge to $f$ $\mu$-a.s. Indeed, it would exist then a linear Borel subspace $E_1$ of $E_0$ such that $\lambda(E_1) = 1$ (resp. $\rho(E_1) = 1$) and $f_k(e) \to f(e)$ for every $e$ from $E_1$. It would follow that $(f_k)$ would converge to $f$ $\mu$-a.s. Contradiction.

**Remark.**

To see that Proposition 2 and 3 fulfill what was advertised in the Introduction is more complicated than in the case of Proposition 1. This will be included in a much longer paper which is now under preparation.

**REFERENCES.**


ON $p$-LATTICE SUMMING AND $p$-ABSOLUTELY SUMMING OPERATORS

J. Szulga

Abstract. Banach spaces $E$ and Banach lattices $X$ are investigated such that every $p$-absolutely summing operator from $E$ into $X$ is $p$-lattice summing.

Introduction. $p$-absolutely summing operators have some natural analogues if the rank space is a Banach lattice. An operator $T$ from a Banach space $E$ into a Banach lattice $X$ is called $p$-lattice summing if for every sequence $(x_n)$ in $E$ such that $\sum |\langle x'_n, x_n \rangle|^p < \infty$ for all $x' \in E'$, the "series" $\left( \sum |T x_n|^p \right)^{1/p}$ converges in $X$. These operators were studied in [13], [12], [7]. In particular one can find in [7] a characterization of all pairs $(E, X)$ such that $\Lambda_p(E, X) \subseteq \Pi_p(E, X)$, where $\Lambda_p$ and $\Pi_p$ denote the suitable spaces of $p$-lattice and $p$-absolutely summing operators. The converse inclusion is almost evident for $p=1$ or $p=2$ and false for $p \in (1, 2)$ ([7]).

In this paper we give some geometric conditions for $\Pi_p(E, X) \subseteq \Lambda_p(E, X)$ to be hold true, $1 < p < 2$. However, in some extreme cases our knowledge is not satisfactory and the problem remains open.

Notations. Throughout the paper $E$, $F$, ... denote Banach spaces and $X$, $Y$, ... denote Banach lattices. $E'$ is the topological dual of $E$. If $p \in (1, \infty)$ then $p'$ denotes the dual exponent of $p$, i.e. $1/p + 1/p' = 1$. J. Krivine ([1]) introduced a calculus of 1-homogeneous functions on $X^n$ which makes possible to investigate expressions like $\left( \sum |x_i|^p \right)^{1/p}$, where $x_1, \ldots, x_n \in X$. We put

$$(\sum |x_i|^p)^{1/p} = \sup \left\{ \frac{\sum a_i x_i}{\sum |a_i|^p} : a_i \in \mathbb{R}, \sum |a_i|^p' \leq 1 \right\}.$$

An operator $T : E \to X$ is said to be $p$-lattice summing if there is a $C > 0$ such that

$$\| \left( \sum |T x_i|^p \right)^{1/p} \| \leq C \sup \left\{ \left( \sum |\langle x'_i, x_i \rangle|^p \right)^{1/p} : x' \in E', \|x'\| \leq 1 \right\}$$

for all finite sequences $x_1, \ldots, x_n \in E$. 
Recall that an operator $T : E \to F$ is $p$-absolutely summing if there is a $C > 0$ such that

$$\left( \sum \|Tx_i\|^p \right)^{1/p} \leq C \sup \left\{ \left( \sum \|x'_i, x_i\|^p \right)^{1/p} : x' \in E', \|x'\| \leq 1 \right\}$$

for all finite sequences $x_1, \ldots, x_n \in E$. We denote by $\lambda_p(T)$ and $\tau_p(T)$ respectively, the smallest constants in the above inequalities which turn the suitable classes $\Lambda_p(E, X)$ and $\Pi_p(E, F)$ of $p$-summing operators into Banach spaces. Let us note that $\infty$-lattice summing operators are exactly Schlotterbeck's majorizing operators ([10]), hence $T \in \Lambda_\infty(E, X)$ if and only if there exists a factorization $T = UV$, where $V : E \to C(K)$, $\|V\| \leq \lambda_\infty(T)$ and $U : C(K) \to X$, $\|U\| \leq 1$ ([10]).

In order to avoid some technical difficulties we assume that $X$ is complemented in the bidual $X''$. This restriction is not essential for our purpose.

An operator $T : E \to X$ is said to be $p$-integral if $T = UV$, where $V \in \Lambda_\infty(E, L_p^\infty)$ and $U : L_p \to X$ is an arbitrary operator (cf[8]).

We say that a Banach space $E$ is of cotype $q$, $2 \leq q < \infty$, if there is a $C > 0$ such that

$$\left( \sum \|x_i\|^q \right)^{1/q} \leq C \int_0^1 \|\sum r_i(t)x_i\| \, dt$$

for all finite sequences $x_1, \ldots, x_n \in E$. Here $r_i$ denote Rademacher functions. A Banach lattice $X$ is said to be $p$-convex, if there is a $C > 0$ such that

$$\| \left( \sum |x_i|^p \right)^{1/p} \| \leq C \left( \sum \|x_i\|^p \right)^{1/p}$$

for all finite sequences $x_1, \ldots, x_n \in X$.

By definition, $E$ is finitely representable in $F$ if there is a $C \geq 1$ such that every finite dimensional subspace of $E$ is $C$-isomorphic to a subspace of $F$. Similarly we define lattice finite representability just taking a suitable lattice isomorphism.

For the above and other concepts we refer to [3] and [8].

**Main results.** We start from two characterizations of $p$-lattice summing operators. The following results are taken from [7].

**Theorem 1.** An operator $T : E \to X$ is $p$-lattice summing if and only if for any operator $S : l_p \to E$, $TS$ is $\infty$-lattice summing.

**Theorem 2.** Let $p = 1$ or $p = 2$. An operator $T : E \to X$ is $p$-lattice summing if and only if for any positive operator $U : X \to L_1$, $UT$ is $p$-absolutely summing.
Hence for $p = 1$ or $p = 2$ $\Pi_p(E, X) \subseteq \Lambda_p(E, X)$. The inclusion is false for $p \in (1, 2)$.

**Proposition 3.** Let $1 < p < 2$. Let $X$ be rearrangement invariant function space on $[0, 1]$ which contains a function $f$ with $p$-stable distribution. Then there exists an operator $T : c_0 \to X$ such that $\pi_p(T) < \infty$ and $\lambda_p(T) = \infty$.

**Proof:** Let $b = (b_i) \in 1_p$. Define $U : c_0 \to 1_p$ by $U(a_i) = (a_i b_i)$ and $V : 1_p \to X$ by $V(c_i) = \sum_i c_i f_i$, where $f_i$ are independent copies of $f$. Then $\|U\| = \|b\|_p$ and $\|V\| = \|f\|_X$. We have taking $T = VU$,

$$\pi_p(T) \leq \|U\| \pi_p(U) = \|U\| \|V\| < \infty.$$ 

Now

$$\lambda_p(T) = \sup \{ \| (\sum_i \sum_k a_{ik} f_i b_i)^{1/p} \|_X : \sup \{ \sum_i |a_{ik}|^{1/p} \leq 1 \} \}
\leq \sup \{ \| (\sum_i |z_i f_i b_i|_p) \|_X : \sup \|z_i\|_p \leq 1 \} \leq \sup \{ \| (\sum_i z_i f_i b_i) \|_X : \sup \|z_i\|_p \leq 1 \}
= \sup \{ \| (\sum_i z_i f_i b_i) \|_X : \sup \|z_i\|_p \leq 1 \}
= \infty$$

by [11].

As examples of such $X$ we can consider $L_q(0, 1)$, $1 \leq q < p$ or the Orlicz spaces $L_M(0, 1)$, where $M(t) = t^p / |\log t|^a(t \to \infty)$, $a > 1$.

Now the general problem can be formulated:

**Problem.** Describe all pairs $(E, X)$ such that $\Pi_p(E, X) \subseteq \Lambda_p(E, X)$, where $1 < p < 2$.

The case $p > 2$ also can be taken under the consideration. A partial solution is contained in the following result. Let $q_E = \inf\{q : E$ is of cotype $q\}$.

**Theorem 4.** Let $X$ be as in Proposition 3. If $1 < p < 2$ then $\Pi_p(E, X) \subseteq \Lambda_p(E, X)$ if and only if $q_E' > p$.

**Proof:** Let $q_E' > p$. Then by [5] $\Pi_p(E, X) \subseteq \Pi_1(E, X)$ and $\Pi_1(E, X) \subseteq \Lambda_p(E, X)$ by [7].

Assume now $\Pi_p(E, X) \subseteq \Lambda_p(E, X)$ with the embedding constant $C$. Then

$$\lambda_p(JS) \leq c_p \lambda_\infty(S)$$

for any operator $S : E \to 1_p$.

where $J$ is the natural embedding from $1_p$ into $X$, i.e. $J(a_i) = \sum_i a_i f_i$ and $f_i$ are independent copies of $f$, $c_p = C \|f\|_X$. In fact, if
$S = \sum x'_i \otimes e_i$, where $x'_i \in E'$ and $(e_i)$ is the standard basis of $l_p$
then we have putting $T = JS$

$$
\pi_p(T) = \sup_{(x_i^*, x_k)} \left( \sum_i \| x'_i, x_k \| f_i \|_X \right)^{1/p} \sup_{\| x^* \|_1} \left( \sum_i \| x'_i, x_k \|_P \right)^{1/p} \\
= \| f \|_X \sup_i \left( \sum_i \| x'_i \|_P \right)^{1/p} \| U \| : U : E' \to l_p \\
\leq \| f \|_X \left( \sum_i \| x'_i \|_p \right)^{1/p} \\
= \| f \|_X \lambda_\infty(S).
$$

Since majorizing operators have the extension property ([10]), thus (*) carries to any subspace of $E$, moreover to any Banach space which is finitely representable in $E$. So we can take $l_{q_E}$ instead of $E$ in (*).

Since (cf the proof of Proposition 3)

$$
\lambda_p(E) \geq \left\{ \sum_i \| U x'_i f_i \|_1 \right\}^{1/p}
$$

for all $U : E' \to l_p$ with $\| U \| \leq 1$, hence we arrive at contradiction if we assume $q'_E \leq p$. To see this it suffices to take the embedding map $U : l_{q'_E} \to l_p$ and the orthogonal normed $x'_i \in l_{q'_E}$. Then by L. Schwartz theorem [11]

$$
c (n \log n)^{1/p} \leq \left\{ 1 \sum_i \| f_i \|_p \right\}^{1/p} = \left\{ 0 \sum_i \| U x'_i f_i \|_1 \right\}^{1/p} \\
\leq c_p \left( \sum_i \| x'_i \|_1 \right)^{1/p} q'E \leq c p n^{1/p}
$$

Therefore $q'_E > p$.

**Corollary.** The following assertions are equivalent:

1° (*) ,

2° all operators $U : E' \to l_p$ are p-stable,

3° all operators $U : E' \to l_{q'}$ are p-stable for some (each) $q \leq p$, $q' > p$.

**Proof:** By Theorem 4 1° $\iff$ 4°. Since 2° is the special case of 1°, namely for $X = L_1$, hence 2° $\iff$ 4°. Since $l_p$ embeds into $L_1$, thus 3° $\Rightarrow$ 2°. Finally, 4° implies $\Pi_p(E, l_p) \leq \Pi_1(E, l_1) \leq \Lambda_q(E, l_1)$ which yields 3°.

One should note that 4° means exactly that all operators from $E$ into $l_p$ are p-stable (cf[9]). Moreover, by Maurey's result([4]), if $E'$ is a Banach lattice of finite cotype and $F$ is not-p-stable Banach space then the property "all operators from $E'$ into $F$ are p-stable" yields $E'$ to be p-stable itself (the assumption excludes the case $E = l_1$).
It is clear that for $p$-convex $X$, $\prod_p (B_p X) \subseteq \Lambda_p (B_p X)$. Now we ask whether the converse holds in the case $q, p \leq p$. Unfortunately we are not able to answer this question even taking "simple" spaces e.g. $B_p = l_q$, $p' \leq q \leq \infty$. We are going to give some equivalent conditions to this under the question. Let $S_p$ be a subspace of $L_1$ spanned by independent $p$-stable functions, we denote by $I_p$ the ideal of $p$-integral operators.

**Theorem 5.** The following properties of $X$ are equivalent:

1\textsuperscript{o} $\prod_p (c_0 X) \subseteq \Lambda_p (c_0 X)$,

2\textsuperscript{o} $I_p (l_p X) \subseteq \Lambda_p (l_p X)$,

3\textsuperscript{o} all operators from $l_p$ into $X$ are $p$-convex,

4\textsuperscript{o} there is a $C > 0$ such that

$$\| \sum g_i x_i \| \leq C (\sum \| g_i \|)^{1/p} \sup (\sum \langle x_i, x_i' \rangle)^{1/p'}$$

for all finite sequences $x_1, \ldots, x_n \in X$ and $g_1, \ldots, g_n \in S_p$.

**Proof.** Since on $L_1$-space $I_p = I_p$ hence by Theorem 1 the assertion 1\textsuperscript{o} says that any operator of the form

$$W = V \in \mathcal{L}_1 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_p \rightarrow X$$

is majorizing, which is exactly what 2\textsuperscript{o} means. By duality 2\textsuperscript{o} is equivalent to the property:

if $T \in \Lambda_\infty (l_p X)$ then $T' \in \prod_p (X, l_p')$

(cf [6]). Applying the latter to $T$ such that $T e_i = x_i'$, where $(e_i)$ is a basis of $l_p$ we infer that all operators from $X'$ into $l_p$ are $p'$-concave, which in turn gives 3\textsuperscript{o} (cf [4]). In fact,

$$\pi_p (x') = \sup \| (\sum |x_1| x_i')^{1/p'} \|$$

and

$$\lambda_\infty (T) = \| (\sum |x_1| x_i')^{1/p'} \|$$

Finally, 4\textsuperscript{o} is just reformulation of (say) 3\textsuperscript{o} in terms of finite rank operators where classical properties of $p$-stable functions are used.

**Concluding remarks and problems.** The condition "$\prod_p (l_p X) \subseteq \Lambda_p (l_p X)$" is stronger than any of 1\textsuperscript{o} - 4\textsuperscript{o}. It says exactly that $\prod_p (B_p X) \subseteq \Lambda_p (B_p X)$ for every Banach space $B$. This is the consequence of the fact that $T : E \rightarrow X$ is $p$-absolutely summing if and only if $T S$ is such for all


The evident sufficient condition for the above property (and hence for $1^0 - 4^0$) is $p$-convexity of $X$. On the other hand we infer that any of $1^0 - 4^0$ yields $p_X = \sup \{ r : X \text{ is } r\text{-convex} \} \geq p$. In fact, were $p_X < p$ we could replace in (say) $1^0$ $X$ by $L_p(0,1)$(which contains $p$-stable functions) so by Proposition 3 we arrive at contradiction.

Problem. Whether exists $X$, $p_X = p$ not $p$-convex but still all operators $T : l_p \to X$ are $p$-convex?

The subject we deal with is related to problems investigated in [2] as $p$-lattice summing operators are Banach lattice versions of $\Theta_p$-Radonifying operators which form slightly smaller class.

Note that the condition $3^0$ of Theorem 5 can be written in the form:

$$\| (\sum |T x_i|^p)^{1/p} \| \leq C \| T \| \| (\sum |x_i|^p)^{1/p} \|, \quad T : l_p \to X, \quad (x_i) \subseteq l_p$$

N. Nielsen observed([6])that there is a Banach lattice $X$ with $p_X < p$ which contains a subspace $E$ isomorphic to $l_p$ such that

$$\| (\sum |T x_i|^p)^{1/p} \| \leq C \| T \| \| (\sum |x_i|^p)^{1/p} \|, \quad (x_i) \subseteq E$$

for all $T : E \to X$. Note that (+) and (++) differ essentially since the lattice structure of $X$ instead of $E$ is involved into the calculation of the expressions on the right hand side. Since the Nielsen's example consists of the Orlicz space $L^{M}_{\infty}(0,1)$ where $M(t) = t^p/|\log t|^a$ with $a > 1$, thus we conject that $L^{M}_{\infty}(0,1)$ is a possible candidate for the solution of the problem above.

References


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NOTE ON CHUNG-TEICHER TYPE CONDITIONS FOR THE STRONG LAW OF LARGE NUMBERS IN A HILBERT SPACE

D. Szynal and A. Kuczmaszewska

1. INTRODUCTION. Let $H$ be a separable Hilbert space with a scalar product $\langle \cdot , \cdot \rangle$. A strongly measurable mapping $X$ from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $H$ is said to be a random element. If $\mathbb{E} \| X \| < \infty$, then the expectation $\mathbb{E}X$ is defined by Bochner integral.

Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a continuous, even and non-decreasing on $(0, \infty)$ function with $\lim_{x \to \infty} \varphi(x) = \infty$ such that

(a) $\varphi(x)/x^+ \text{ or } (b) \varphi(x)/x^+, \varphi(x)/x^{2+} \text{ as } x \to \infty$.

Suppose that $(X_n, n \geq 1)$ is a sequence of independent random elements taking values in a real, separable Hilbert space with $\mathbb{E}X_n = 0$, $n \geq 1$, and such that $\sum_{n=1}^{\infty} \mathbb{E}\varphi(\|X_n\|)/\varphi(n) < \infty$. From results of [4] it follows that then $n^{-1} \sum_{k=1}^{n} \|X_k\| \to 0 \text{ a.s., } n \to \infty$, i.e. Kai-Lai Chung's [1] strong law of large numbers (SLLN).

It was proved by H. Teicher [3] that if $(X_n, n \geq 1)$ is a sequence of independent random variables with $\mathbb{E}X_n = 0$, $n \geq 1$,

\[ \sum_{j=2}^{\infty} \frac{\mathbb{E}X_j^2}{j^a} < \infty, \]
\[ \sum_{j=2}^{\infty} \frac{\mathbb{E}X_i^2}{j} < \infty, \]
\[ \sum_{k=1}^{\infty} \frac{\mathbb{E}X_k^2}{n} = o(1) \text{ as } n \to \infty, \]
\[ \sum_{n=1}^{\infty} P(|X_n| \geq a_n) < \infty \]

for some positive numerical sequence $(a_n, n \geq 1)$ with

\[ \sum_{n=1}^{\infty} a_n^2 \mathbb{E}X_n^2 < \infty \text{ then SLLN holds.} \]

The aim of this note is to prove SLLN of Kai-Lai Chung and Teicher type in the case when $(X_n, n \geq 1)$ is a sequence of independent random
elements taking values in a real, separable Hilbert space $H$. The presented result contains at the same time an extension of some Woyczyński's result [4].

2. SLLN OF KAI-LAI CHUNG AND TEICHER TYPE.

THEOREM. Let $\{X_n, n \geq 1\}$ be a sequence of independent random elements from $H$. Suppose that in the case (a)

$$\begin{align*}
(A) & \quad \sum_{j=2}^{\infty} \frac{\varphi^2(\|X_j\|)}{\varphi(j) + \varphi^2(\|X_j\|)} \sum_{i=1}^{j-1} \frac{\varphi^2(\|X_i\|)}{\varphi(i) + \varphi^2(\|X_i\|)} < \infty, \\
(B) & \quad n^{-2} \sum_{i=1}^{n} i^2 \sum_{j=1}^{\infty} \frac{\varphi^2(\|X_j\|)}{\varphi(i) + \varphi^2(\|X_i\|)} = o(1), \\
(C) & \quad \sum_{n=1}^{\infty} P(\|X_n\| < a_n) < \infty
\end{align*}$$

for some positive numerical sequence $\{a_n, n \geq 1\}$ with

$$\begin{align*}
(D) & \quad \sum_{n=1}^{\infty} \varphi^2(a_n) E\left[\frac{\varphi^2(\|X_n\|)}{\varphi^2(a_n) + \varphi^2(\|X_n\|)}\right] < \infty
\end{align*}$$

or in the case (b)

$$\begin{align*}
(A') & \quad \sum_{j=2}^{\infty} \frac{\varphi(\|X_j\|)}{\varphi(j) + \varphi(\|X_j\|)} \sum_{i=1}^{j-1} \frac{\varphi(\|X_i\|)}{\varphi(i) + \varphi(\|X_i\|)} < \infty, \\
(B') & \quad n^{-2} \sum_{i=1}^{n} i^2 \sum_{j=1}^{\infty} \frac{\varphi(\|X_j\|)}{\varphi(i) + \varphi(\|X_i\|)} = o(1), \\
\end{align*}$$

and (C) is satisfied for some positive numerical sequence $\{a_n', n \geq 1\}$ with

$$\begin{align*}
(D') & \quad \sum_{n=1}^{\infty} \varphi(a_n') E\left[\frac{\varphi(\|X_n\|)}{\varphi^2(a_n') + \varphi^2(\|X_n\|)}\right] < \infty.
\end{align*}$$

Then

$$\begin{align*}
(1) & \quad \sum_{k=1}^{n} (X_k - E_k I_{\{\|X_k\| < k\}}) \to 0 \quad a.s., \quad n \to \infty.
\end{align*}$$

PROOF: Set $X'_n = X_n I_{\{\|X_n\| < n\}}$ and $X'_n = X'_n - E X'_n$.

Write
(2) \( n^{-2} \sum_{i=1}^{n} x_i^2 = n^{-2} < \sum_{i=1}^{n} x_i', \sum_{i=1}^{n} x_i'^2 > = n^{-2} \sum_{i=1}^{n} x_i', x_i'^2 + 2n^{-2} \sum_{i=1}^{n} x_i', x_i'^2 > .

Note now that in the case (a) \( \|X'_n\| / n \leq \varphi (\|X'_n\|) / \varphi (n) \)
while in the case (b) \( \|X'_n\|^2 / n^2 \leq \varphi (\|X'_n\|) / \varphi (n) \).

Hence, putting

\[ Y_n = \|X'_n\|^2 \mathcal{I}[\|X'_n\| \leq a_n], \quad Z_n = Y_n - \mathbb{E}Y_n, \]

where \( \{a_n, n \geq 1\} \) is a sequence of positive numbers, we get in the case (a)

\[
\sum_{n=1}^{\infty} \mathbb{E}(Z_n/n^2)^2 \leq \sum_{n=1}^{\infty} \mathbb{E}(Y_n^2/n^4) = \sum_{n=1}^{\infty} n^{-4} \mathbb{E} \|X'_n\|^4 \mathcal{I}[\|X'_n\| \leq a_n] \leq
\]

\[
\leq 8 \sum_{n=1}^{\infty} n^{-4} \mathbb{E} \{ (\|X'_n\|^4 + \mathbb{E}^n \|X'_n\|) \mathcal{I}[\|X'_n\| \leq a_n] \} =
\]

\[
= 8 \sum_{n=1}^{\infty} n^{-4} \mathbb{E} \|X'_n\|^4 \mathcal{I}[\|X'_n\| \leq a_n] + 8 \sum_{n=1}^{\infty} n^{-4} \mathbb{E} \|X'_n\| \mathcal{E} \mathcal{I}[\|X'_n\| \leq a_n] =
\]

\[
= 8 \sum_{n=1}^{\infty} n^{-4} \mathbb{E} \|X'_n\|^4 \mathcal{I}[\|X'_n\| \leq a_n] + 8 \sum_{n=1}^{\infty} n^{-4} \mathbb{E} (\|X'_n\|^4 \mathcal{I}[\|X'_n\| \leq a_n]) +
\]

\[
+ 8 \sum_{n=1}^{\infty} n^{-4} \mathbb{E} (\|X'_n\|^4 \mathcal{I}[\|X'_n\| > a_n]) \leq 16 \sum_{n=1}^{\infty} n^{-4} \mathbb{E} \|X'_n\|^4 \mathcal{I}[\|X'_n\| \leq a_n] +
\]

\[
+ 8 \sum_{n=1}^{\infty} \mathbb{P}[\|X'_n\| > a_n] \leq 16 \sum_{n=1}^{\infty} \mathbb{E} \frac{\varphi^2(\|X'_n\|)}{\varphi^4(n)} \mathcal{I}[\|X'_n\| < \varphi^2(\|X'_n\|) \mathcal{I}[\|X'_n\| \leq a_n]^+ +
\]

\[
+ 8 \sum_{n=1}^{\infty} \mathbb{P}[\|X'_n\| > a_n] \leq 32 \sum_{n=1}^{\infty} \mathbb{E} \frac{\varphi^2(\|X'_n\|)}{\varphi^4(n) + \varphi^2(\|X'_n\|)} \mathcal{I}[\|X'_n\| \leq a_n]^+ +
\]

\[
+ 8 \sum_{n=1}^{\infty} \mathbb{P}[\|X'_n\| > a_n] \leq 32 \sum_{n=1}^{\infty} \varphi^2(a_n) \mathbb{E} \frac{\varphi^2(\|X'_n\|)}{\varphi^4(n) + \varphi^2(\|X'_n\|)} +
\]

\[
+ 8 \sum_{n=1}^{\infty} \mathbb{P}[\|X'_n\| > a_n] < \infty.
\]
In the case (b) we have

\[
\sum_{n=1}^{\infty} \mathbb{E} \left( \frac{Z_n}{n^2} \right)^2 \leq 16 \sum_{n=1}^{\infty} \mathbb{E} \frac{E^n}{n^2} \mathbb{I} \left[ \|X_n\| \leq a_n \right] + 8 \sum_{n=1}^{\infty} P \left[ \|X_n\| > a_n \right] \leq \\
\leq 16 \sum_{n=1}^{\infty} \mathbb{E} \frac{\varphi^2(\|X_n\|)}{\varphi^2(n)} \mathbb{I} \left[ \|X_n\| \leq a_n \right] + 8 \sum_{n=1}^{\infty} P \left[ \|X_n\| > a_n \right] \leq \\
\leq 32 \sum_{n=1}^{\infty} \mathbb{E} \frac{\varphi^2(\|X_n\|)}{\varphi^2(n) + \varphi^2(\|X_n\|)} + 8 \sum_{n=1}^{\infty} P \left[ \|X_n\| > a_n \right] < \infty.
\]

This proves that

\[
\sum_{n=1}^{\infty} \left( \frac{Z_n}{n^2} \right) < \infty \quad \text{a.s.},
\]

which implies by Kronecker's lemma

\[
n^{-2} \sum_{i=1}^{n} (Y_i - \mathbb{E}Y_i) \to 0 \quad \text{a.s., } n \to \infty.
\]

Moreover, we note that in the case (a) the assumption (B) implies

\[
n^{-2} \sum_{i=1}^{n} \mathbb{E}Y_i = n^{-2} \sum_{i=1}^{n} \mathbb{E}X_i' - \mathbb{E}X_i'^2 \mathbb{I} \left[ \|X_i\| \leq a_i \right] \leq 4 n^{-2} \sum_{i=1}^{n} i^2 \mathbb{E}\|X_i\|^2/i^2 \leq \\
\leq 8 n^{-2} \sum_{i=1}^{n} E i^2 \mathbb{E} \frac{\varphi^2(\|X_i\|)}{\varphi^2(i) + \varphi^2(\|X_i\|)} = o(1),
\]

while in the case (b) the assumption (B') implies

\[
n^{-2} \sum_{i=1}^{n} \mathbb{E}Y_i \leq 8 n^{-2} \sum_{i=1}^{n} i^2 \mathbb{E} \frac{\varphi(\|X_i\|)}{\varphi(i) + \varphi(\|X_i\|)} = o(1).
\]

Hence, we conclude that

\[
n^{-2} \sum_{i=1}^{n} Y_i \to 0 \quad \text{a.s., } n \to \infty.
\]

Now taking into account (C), we get

\[
\sum_{n=1}^{\infty} P \left[ Y_n \neq \frac{\mathbb{E}X_n'}{\|X_n\|^2} \right] = \sum_{n=1}^{\infty} P \left[ \|X_n\| > a_n \right] < \infty.
\]
Therefore, in the case (a) or in the case (b)

\[(3) \quad n^{-2} \sum_{i=1}^{n} \|X_i'\|^2 = n^{-2} \sum_{i=1}^{n} \langle X_i', X_i' \rangle \rightarrow 0 \text{ a.s., } n \to \infty.\]

We prove now that

\[n^{-2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \langle X_i', X_j' \rangle \rightarrow 0 \text{ a.s., } n \to \infty.\]

It is not difficult to verify that

\[\sum_{i=2}^{n} \sum_{j=1}^{i-1} \langle X_i', X_j' \rangle \text{ is a martingale with respect to } \sigma\text{-field } \sigma(X_1', X_2', \ldots, X_n'), \quad n \geq 1, \quad \text{and } U_{2i} = \sum_{j=1}^{i-1} \langle X_1', X_j' \rangle\]

is a martingale difference sequence.

Taking into account that $U_{2i}$ takes values in $\mathbb{R}$ we have in the case (a)

\[\sum_{i=2}^{\infty} \mathbb{E} |U_{2i}|^2/i^a \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \mathbb{E} \|X_i' \|^2 \mathbb{E} \|X_j' \|^2 \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \mathbb{E} \|X_i' \|^2 \mathbb{E} \|X_j' \|^2 = \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \mathbb{E} \|X_i' \|^2 \mathbb{E} \|X_j' \|^2 \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} < \infty\]

while in the case (b)

\[\sum_{i=2}^{\infty} \mathbb{E} |U_{2i}|^2/i^a \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} \leq \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{\mathbb{E} \|X_i' \|^2}{i^2} \frac{\mathbb{E} \|X_j' \|^2}{j^2} < \infty\]
\[ \leq 2^6 \sum_{i=2}^{\infty} i^{-2} E \frac{\phi(||X_i||)}{\phi(i) + \phi(||X_i||)} \leq 1 \]

Hence, by [5]

\[ (4) \quad n^{-2} \sum_{i=2}^{n} U_{2i} = n^{-2} \sum_{i=2}^{n} j^{-2} E < \frac{\phi(j)}{\phi(i) + \phi(||X_j||)} \rightarrow 0 \text{ a.s., } n \rightarrow \infty. \]

Therefore, by (2) - (4), we get

\[ n^{-1} \sum_{i=1}^{n} \frac{\phi_{ij}}{\phi(X_i)} = 0 \text{ a.s., } n \rightarrow \infty. \]

Now taking into account the properties of \( \phi \) and the condition (C), we have

\[ \sum_{n=1}^{\infty} P \left[ \|X_n\| \neq n \right] = \sum_{n=1}^{\infty} E \left[ \|X_n\| \geq n \right] = \sum_{n=1}^{\infty} \left[ \|X_n\| > a_n \right] \]

\[ + \sum_{n=1}^{\infty} \left[ \|X_n\| \leq a_n \right] \leq \sum_{n=1}^{\infty} P \left[ \|X_n\| > a_n \right] \]

\[ + 2 \sum_{n=1}^{\infty} \frac{\phi^2(||X_n||)}{\phi^2(n) + \phi^2(||X_n||)} \leq \sum_{n=1}^{\infty} P \left[ \|X_n\| > a_n \right] + 2 \sum_{n=1}^{\infty} \frac{\phi^2(||X_n||)}{\phi^2(n) + \phi^2(||X_n||)} \]

\[ \leq \sum_{n=1}^{\infty} P \left[ \|X_n\| > a_n \right] + 2 \sum_{n=1}^{\infty} \frac{\phi^2(||X_n||)}{\phi^2(n) + \phi^2(||X_n||)} \]

Putting \( r = 2 \) and \( r = 1 \) in the case (a) and (b) respectively, we get

\[ \sum_{n=1}^{\infty} P[X_n \neq X_n] < \infty, \text{ which completes the proof of (1).} \]

COROLLARY. If \( (B) \) and \( (B') \) are replaced by the condition

\[ n^{-1} \sum_{i=1}^{n} E \left( \frac{\phi(||X_i||)}{\phi(i)} \right) = o(1), \text{ then } \{X_n, n \geq 1 \} \text{ satisfies the strong law of large numbers, i.e. } S_n/n \rightarrow 0 \text{ a.s., } n \rightarrow \infty. \]

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STABLE PROCESSES AND MEASURES; A SURVEY*

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§0. Introduction

One of the main motivations for studying probability theory on vector spaces is the possibility of applying the general theory to stochastic processes. In this paper, we report the recent progress in the study of stable processes. It is true that the theory of stable measures on vector spaces and the theory of stable processes are developed rather separately. However, the aim of this survey is to show that there are many similarities and common points between them. It is our hope that this will stimulate more interactions and will help in understanding better the fascinating "stable world" so different from the Gaussian one. Why is this important?

We believe stable distributions and stable processes do provide useful models for many phenomena observed in diverse fields. The most commonly met classical probability distribution functions in the analysis of physical processes are Gaussian or have exponential tails. In recent years, however, long inverse power tails have become more evident. The basic feature of stable distributions is that they form a natural class of limit distributions and they have long tails. Let's mention also that the classical Levy definition of a stable distribution $\mu$ has an immediate physical interpretation as "superposability" property. Namely, for any $A > 0$ and $B > 0$ there exists $C > 0$ and $s \in \mathbb{R}$ such that

$$[\mu * m^{-1}_A * \mu * m^{-1}_B] * t^{-1}_s = \mu * m^{-1}_C,$$

where $m_A(x) = Ax$, $t_s(x) = x + s$. These are reasons for which stable distributions and processes arise in the following situations.

There are many physical phenomena which exhibit both space and time long tails and thus seem to violate the requirement of Gaussian distribution as a limit in the traditional central limit theorem. However, since these physical systems are superposable in the sense a Gaussian is, but without second moments, one suspects the use of stable distributions which have long tails to be relevant in the physics of these phenomena. A clear physical basis is required to justify the use of stable distributions in much the same way Khinchin (1949) gave a physical justification for the use of Gaussian distributions. Tunaley (1972) invoked physical arguments to suggest that if the frequency distributions in metallic films are superposable in the sense of Levy, then the observed noise characteristics in them may be understood. Based only on the experimental observations that near second order phase transitions where long tail spatial order develops, Jona-Lasinio (1975) considered stable distributions as a basic ingredient in understanding renormalization group notions in explaining such phenomena. See also a review article by Cassandro and Jona-Lasinio (1978). A careful mathematical basis of the physics of condensed matter system reveals that
one has a semibounded energy spectrum for thermodynamically stable systems and
the requirement of superposability of these spectra is a consequence of the
physical necessity that a sum of two non-interacting systems must have their
spectrum related to the individual spectra. Rajagopal et al. (1983) suggested
the use of stable limit distributions of low energy excitations in complex
condensed matter systems in conjunction with the Paley-Wiener Theorem to
accomodate the semibounded spectra to explain the observed nonexponential
temporal decay of relaxation processes. By employing some facts on a-stable
density functions, $0 < a < 1$, which are relevant to situations where the
spectral density is one-sided and superposable, and since the time evolution of
such a system is related to the Fourier transform of this density, Weron et al.
(1984) have established the nonexponential behavior of the relaxation functions
in full generality, without invoking the Paley-Wiener theorem as was originally
done by Rajagopal et al. (1983).

It is important to point out that the mathematical result of stable
distributions with proper physical foundations can be of immense value in
discussing complex temporal and spatial properties of condensed matter systems.
However, several authors have studied the mathematical consequences of using
stable processes in some ideal physical contexts. These studies are useful but
require a stronger physical underpinning however in order to be fully appreciated
as a fundamental basis for the observed physical phenomena. Quite often, it is
not enough to recognize the analogy of a physical phenomena with a mathematical
structure. A more detailed physical basis for the analogy gives such an
observation a fuller meaning to the underlying mathematics and to the structure
of physical phenomena.

As examples of the exploration of the mathematical structures in physical
contexts, we may cite a few very interesting papers. Doob (1942), West and
Seshadri (1982) examined the energy of a linear system driven by stable
fluctuations. Mandelbrot and van Ness (1968) used Gaussian and stable fractional
stochastic processes in several interesting situations arising in Economics,
Hydrology, and Physics. (See also Mandelbrot (1982)). Hughes, Shlesinger, and
Montroll (1981) and, Montroll and Shlessinger (1983) examined random walks with
self-similar clusters leading to "Levy flights" and "1/f noise".

A class of stochastic processes with time dependent transition rates having
an $\alpha$-stable relaxation function is considered in Teitler, Rajagopal and Ngai
(1982). The corresponding monomial relaxation Ngai model can be applied in the
analysis of measurements of dielectric response, mechanical response, nuclear
spin-lattice relaxation, and transient transport in materials of widely different
chemical configurations and physical states.

Failure of the least-squares method of forecasting in economic time series
was first explained by Mandelbrot (1963). He introduced a radically new approach based on \( \alpha \)-stable processes to the problem of price variation. Income distributions are mostly log-normal except for the last few percentile which have \( \alpha \)-stable tails with exponents \( \alpha = 1.6 \), see Badgar (1980).

These and related empirical findings suggest three tasks to the probabilist:
(i) to press the development of the general theory of stable measures and processes, (ii) to study in detail many specific simple families of stable processes that could in some way be expected to be typical of what happens in the absence of Gaussian character of a model, and (iii) to present curious properties of stable processes to scientists, engineers and statisticians. The present paper contributes in part to these tasks.

We survey the present stage of the theory of stable processes in §1-§7, including many examples which illustrate presented results. In §8, we prove a so-called correspondence principle - a one-to-one correspondence between stable processes with almost all sample paths belonging to some basic function space \( A(T) \) and stable measures on \( A(T) \). This section is a bridge between stable processes and stable measures on vector spaces. In §9-§14, we consider only some actual topics in the theory of stable measures on vector spaces which are closely related to stable processes. For other results in this theory the readers are referred to the recent monographs by Araujo and Gine (1980) and by Linde (1983).

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§1. Preliminaries

The theory of stable distribution was initiated sixty years ago by P. Levy in 1924 and was completed by Gnedenko and Kolmogorov (1954). Readers interested in the historical development should consult Levy (1937) and Feller (1966).

A probability distribution \( \mu \) on \( \mathbb{R} \) is said to be stable if for any \( A > 0 \) and \( B > 0 \) and independent random variables (r.v.'s) \( X \) and \( Y \) with distribution \( \mu \), there exist \( C > 0 \) and \( s \in \mathbb{R} \) such that
\[
L\text{aw}(C(AX + BY) - s) = \mu.
\]
For every stable distribution there exists a unique constant \( \alpha \in (0,2] \) such that \( C = (A^\alpha + B^\alpha)^{-1/\alpha} \). The constant \( \alpha \) is called the characteristic exponent of the distribution. The distribution \( \mu \) is strictly stable if we can take \( s = 0 \).

In other words a r.v. \( Z \) is strictly \( \alpha \)-stable if for any positive \( A \) and \( B \) and independent r.v.'s \( X \) and \( Y \) with the same distribution as \( Z \)
Another equivalent definition which suggests the role of stable laws as limits of normalized sums of i.i.d. r.v.'s is that for every natural number \( n \) there exist constants \( a_n > 0 \) and \( b_n \) such that

\[
X_1 + X_2 + \ldots + X_n \overset{d}{=} a_n X + b_n
\]

where \( X_1, X_2, \ldots, X_n \) are i.i.d. r.v.'s with the same distribution as \( X \). Only the norming constants \( a_n = n^{1/\alpha} \) are possible and the distribution of \( X \) (or the r.v. \( X \)) is strictly \( \alpha \)-stable if \( b_n = 0 \).

The notion of \( \alpha \)-stable distribution on \( \mathbb{R}^d \) or on an infinite dimensional vector space (see §9) is exactly analogous to the above definition.

Consider a multivariate \( \alpha \)-stable distribution \( \mu \) on \( \mathbb{R}^d \). It was a relatively common belief that \( \mu \) can be characterized by the property that all marginals \( \mu \circ f^{-1} \), where \( f \in (\mathbb{R}^d)' = \mathbb{R}^d \), are univariate \( \alpha \)-stable.

Indeed, if \( \mu \) is \( \alpha \)-stable then \( \alpha \)-stability of the marginals follows easily. The converse holds only if \( \mu \) is strictly stable or if \( \alpha > 1 \). For \( 0 < \alpha < 1 \) a counter example has been given recently by D. Marcus (1983), cf. also Gine and Hahn (1983).

In order for a distribution \( \mu \) on \( \mathbb{R} \) to be \( \alpha \)-stable, it is necessary and sufficient that its characteristic function \( \hat{\mu}(t) = \phi(t) \) be given by

\[
\log \phi(t) = \begin{cases} 
\iota \gamma t - (\sigma |t|)^\alpha (1 - i \beta \text{sign}(t) \tan(\pi \alpha /2)) & \text{if } \alpha \neq 1 \\
\iota \gamma t - \sigma |t| - i \beta (2/\alpha) \log |t| & \text{if } \alpha = 1,
\end{cases}
\]

where \( \alpha, \beta, \gamma \) and \( \sigma \) are real constants with \( \sigma > 0 \), \( 0 < \alpha < 2 \) and \( |\beta| < 1 \). Cf. Gnedenko, Kolmogorov (1954). Here \( \alpha \) is the characteristic exponent, \( \gamma \) and \( \sigma \) merely determine location and scale. The choice \( \gamma = 0 \) implies that the stable distribution has expectation equal to zero (if it is finite). In case \( \beta = 0 \) the distributions are symmetric and when \( |\beta| = 1 \) are commonly called completely asymmetric. Only in the case \( 0 < \alpha < 1 \) are the \( \alpha \)-stable distributions with \( |\beta| = 1 \) one-sided, i.e., their support is \([0, \infty)\) in case \( \beta = 1 \) and \((-\infty, 0]\) in case \( \beta = -1 \), see Feller (1966) p. 542. We shall consider only stable distributions with \( \gamma = 0 \) and frequently with \( \beta = 0 \). In such a case the symmetric \( \alpha \)-stable (SaS) distribution has its characteristic function of the special simple form.

A real valued r.v. \( \Theta \) is SaS of parameter \( \sigma \) if for each \( t \in \mathbb{R} \)
and a complex valued r.v. \( \Theta \) is symmetric \( \alpha \)-stable of parameter \( \alpha \) if for all \( z \in \mathbb{C} \)

\[
\mathbb{E} \exp i \Theta = \exp(-c^\alpha |z|^\alpha),
\]

where we write \( \Theta = U + iV \), \( U, V \) real.

It is well known that a real valued \( \alpha \)-stable random variable \( X \) can be written as a product of two independent r.v.'s one of which is a Gaussian \( G \) and another a positive \( \alpha/2 \)-stable r.v. \( \Theta \) independent of \( G \) (Feller (1966) p. 172)

\[
X = \Theta^{1/2} \cdot G.
\]

Moreover, if \( X \) is a complex valued \( \alpha \)-stable r.v. then there exist independent Gaussian variables \( G_1, G_2 \) with mean zero and variance \( \sigma^2 \) and a positive \( \alpha/2 \)-stable r.v. \( \Theta \) independent of \( G_1 \) and \( G_2 \) such that

\[
X = \Theta^{1/2} \cdot (G_1 + iG_2).
\]

If \( \alpha = 2 \), then in this case \( \Theta^{1/2} = \sqrt{2} \) and we have a concordance with the standard definition of a real and complex Gaussian variable. However only for \( \alpha = 2 \), are the real and imaginary parts of a complex \( \alpha \)-stable r.v. independent.

The classical Levy results on the tail behaviour of \( \alpha \)-stable distributions can be stated as follows

\[
\lim_{u \to \infty} u^{-\alpha} \mathbb{P}(|X| > u) = (c(\alpha)\sigma)^\alpha
\]

for a real valued \( \alpha \)-stable r.v. and

\[
\lim_{u \to \infty} u^{-\alpha} \mathbb{P}(|X| > u) = (c(\alpha)r(\alpha)\sigma)^\alpha
\]

for a complex valued \( \alpha \)-stable r.v., where

\[
c(\alpha) = \left[ \int_0^\infty \frac{\sin V}{V^\alpha} \, dv \right]^{-1/2} \quad \text{and} \quad r(\alpha) = \left( \mathbb{E}|c_1^2 + c_2^2|^{\alpha/2} / \mathbb{E}|c_1|^2 \right)^{1/2}.
\]

It follows that \( \mathbb{E}|X|^r < \infty \) for each \( r < \alpha \) for real (or complex) \( \alpha \)-stable r.v.'s and \( \mathbb{E}|X|^r = \infty \) for all \( r > \alpha \).
The real r.v.'s \( X_1, X_2, \ldots, X_n \) are jointly SaS if the real random vector \( X = (X_1, X_2, \ldots, X_n) \) has SaS distribution on \( \mathbb{R}^n \) or equivalently if their joint characteristic function is of the form

\[
\phi_X(t) = \mathbb{E}\exp\{i\langle t, X \rangle\} = \exp\{-\int_{S_n} \langle t, x \rangle^\alpha \, d\Gamma_X(dx)\},
\]

where \( t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \), \( \langle t, x \rangle = t_1x_1 + \cdots + t_nx_n \) and \( \Gamma_X \) is a symmetric finite measure on the unit sphere \( S_n \) of \( \mathbb{R}^n \), called the spectral measure of \( X \). There is a one-to-one correspondence between the distribution of \( X \) and its spectral measure \( \Gamma_X \).

A complex random vector \( X = X_1 + iX_2 \) is SaS if \( X_1, X_2 \) are jointly SaS and then its characteristic function is written with \( t = t_1 + it_2 \), as

\[
\phi_X(t) = \mathbb{E}\exp(i\text{Re}\{t^*X\}) = \mathbb{E}\exp(i\langle t_1X_1 + t_2X_2 \rangle) = \\
\exp\{-\int_{S_2} |t_1x_1 + t_2x_2|^\alpha \, d\Gamma_{X_1, X_2}(x_1, x_2)\}.
\]

When \( X = X_1 + iX_2 \) and \( Y = Y_1 + iY_2 \) are jointly SaS and \( 1 < \alpha < 2 \) the covariation of \( X \) with \( Y \) is defined in Cambanis, Miller (1981) and Cambanis (1983) by

\[
[X, Y]_\alpha = \int_{S_4} \langle x_1 + ix_2, y_1 + iy_2 \rangle^{\alpha-1} \, d\Gamma_{X_1, X_2, Y_1, Y_2}(x_1, x_2, y_1, y_2),
\]

where for a complex number \( z \) and \( a > 0 \) we use throughout the convention

\[
z^{\alpha} = |z|^{\alpha-1} \, z^*,
\]

where \( z^* \) is the complex conjugate of \( z \). The covariation is not generally symmetric, but has some properties analogous to those of the covariance, to which it reduces when \( \alpha = 2 \). Moreover, it introduces a useful concept of semi-inner product on the linear space of jointly SaS r.v.'s. The properties of covariation are as follows:

(i) \( [X_1 + X_2, Y]_\alpha = [X_1, Y]_\alpha + [X_2, Y]_\alpha \)

(ii) \( [ax, by]_\alpha = ab^{\alpha-1}[X, Y]_\alpha \)

(iii) \( [X, Y]_\alpha = 0 \) if \( X \) and \( Y \) are independent.

(iv) \( [X_1, Y_1 + Y_2]_\alpha = [X_1, Y_1]_\alpha + [X_1, Y_2]_\alpha \) if \( Y_1 \) and \( Y_2 \) are independent.
(v) \( \|X\|_\alpha = [X,X]^{1/\alpha} \) is a norm on a linear space of \( \text{SaS} \) r.v.'s equivalent to convergence in probability.

(vi) \( |[X,Y]_\alpha| < \|X\|_\alpha \|Y\|_\alpha^{\alpha-1} \)

(vii) If \( [X,Y]_\alpha = 0 \), then \( Y \) is James orthogonal to \( X \), i.e.,
\[ \|aX + Y\|_\alpha > \|Y\|_\alpha \quad \text{for each } a. \]

In the real case the norm
\[ \|X\|_\alpha = c(\alpha,p) (\mathbb{E}|X|^p)^{1/p} \]
for \( 1 < p < \alpha \), where the constant \( c(\alpha,p) \) depends only on \( \alpha \) and not on \( X \). Also the norm \( \|X\|_\alpha \) determines the distribution of \( X \) via
\[ \phi_x(t) = \exp(-\|X\|^\alpha t)^\alpha \]
This is no longer valid when \( X \) is complex, in which case the \( \|\cdot\|_\alpha^p \)-norm is only equivalent to the \( \|\cdot\|_\alpha^p \)-norm and \( \|X\|_\alpha \) determines only the total mass of the spectral measure via
\[ \|X\|_\alpha = \Gamma(\text{Re}X, \text{Im}X, S_2). \]
For details, see Cambanis (1983).

A stochastic process \( \{X_t, t \in T\} \) is \( \alpha \)-stable if all finite dimensional distributions of the process \( \{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\} \) are \( \alpha \)-stable on \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)). In particular, \( \{X_t, t \in T\} \) is \( \text{SaS} \) iff all linear combinations
\[ a_1 X_{t_1} + \ldots + a_n X_{t_n} \]
are \( \text{SaS} \). It follows from the fact that multivariate distributions are \( \text{SaS} \) iff all marginal distributions are \( \text{SaS} \).

If \( \{Z_t, t \in \mathbb{R}\} \) is a complex \( \text{SaS} \) process with independent increments, then the control measure \( F \) of \( Z \) is defined by
\[ F((s,t]) = \|Z_t - Z_s\|_\alpha^\alpha. \]
In this study integrals of the form \( \int f(\lambda) dZ^\lambda_\alpha \) play a fundamental role. These can be define for any \( f \in L^\alpha(F) \) such that the \( \text{SaS} \) r.v. \( X = \int f(\lambda) dZ^\lambda_\alpha \) has the property
\[ \|X\|_\alpha = \int_\alpha f(\lambda) dZ^\lambda_\alpha = \int f(\lambda) \|d\lambda\|_\alpha^{\alpha}. \]
Moreover, if \( X = \int f(\lambda) dZ^\lambda_\alpha \) and \( Y = \int g(\lambda) dZ^\lambda_\alpha \), where \( f, g \in L^\alpha(F) \), then one has the following:

(i) \( [X,Y]_\alpha = \int f(\lambda) g(\lambda)^{\alpha-1} \|d\lambda\| \)
(ii) $X$ and $Y$ are independent iff $|f||g| = 0$ $F$-a.s.

(iii) $Y$ is James-orthogonal to $X$ iff $\int fg^{<a-1>}F(d\lambda) = 0$.

§2. Basic Classes of Stable Process

A. $\alpha$-stable motion

Let $X$ be r.v. with $\alpha$-stable distribution $F(\cdot, \alpha, \beta)$, $0 < \alpha < 2$, $|\beta| < 1$. Then $(X_t)_{t \geq 0}$ is called $\alpha$-stable (or Levy stable) motion if $X(0) = 0$, $(X_t)_{t \geq 0}$ has homogeneous and independent increments and $X_t$ is distributed as $t^{1/\alpha}X$ if $\alpha \neq 1, 2$, or as $tX + 2/\pi\beta\log t$ if $\alpha = 1$.

Let $D[0,\infty)$ be the set of real-valued functions on $[0,\infty)$ which are right-continuous and have finite left-hand limits. Then there exists a version of $(X_t)_{t \geq 0}$ with all sample paths in $D[0,\infty)$. (Breiman (1968)). Observe that $\alpha = 2$ corresponds to the case of the Wiener process and one may construct the corresponding $\alpha$-stable measure $P_{\alpha, \beta}$ (with $0 < \alpha < 2$ and $|\beta| < 1$) on $D[0,\infty)$ just as Wiener measure is constructed on $C[0,\infty)$.

EXAMPLE.

Consider $D[0,1]$ with the Skorohod topology defined by the convergence in the metric

$$d(x,y) = \inf \{ \sup_{t} |x(t) - y(\lambda(t))| + \sup_{t} |t - \lambda(t)|, \lambda \in \Lambda \},$$

where $\Lambda$ is the class of strictly increasing, continuous mappings of $[0,1]$ into itself. Let $X_1, X_2, \ldots$ be i.i.d. with the common distribution $F(\cdot, \alpha, \beta)$. Define the sequence of r.v.'s $X_n(t)$ in $D[0,1]$ by

$$x_n(t) = \begin{cases} n^{-1/\alpha}(X_1 + \ldots + X_{[nt]}) & \text{if } \alpha \neq 1 \\ n^{-1}(2/(\pi\beta[nt] \log n) - X_1 + \ldots + X_{[nt]}) & \text{if } \alpha = 1 \end{cases}$$

By Skorohod (1957) the distribution of $X_n$ converges weakly under above topology to a limit and this limit coincides with the $\alpha$-stable measure $P_{\alpha, \beta}$.

EXAMPLE

For $0 < \alpha < 1$ and $\beta = 1$ the $\alpha$-stable motion can be constructed with the aid of a Poisson process $Y_t$ with parameter $\lambda$. For this let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with the common distribution function $F(\cdot)$ satisfying the condition
\[ \lambda f(x) = \alpha(1 - \alpha) \sin(\pi \alpha / 2) \cdot x^{-1 - \alpha} dx. \]

Define

\[ X_t = X_1 + \cdots + X_t. \]

In other words, \( X_t \) is a compound Poisson process having the jump of height \( X_n \) at the jump point \( T_n \) of the Poisson process \( Y_t \). Consequently, the sample paths of \( X_t \) are non-decreasing pure jumps functions. Thus \( X_t \) has only upward jumps and between two successive jumps the sample paths are constant. By evaluating \( \varepsilon u X_t \), one can check that it is the \( \alpha \)-stable motion with \( 0 < \alpha < 1 \) and \( \beta = 1 \), see for details (Breiman 1968).

**EXAMPLE**

For \( \alpha = 1/2 \) and \( \beta = 1 \) the \( \alpha \)-stable motion can be obtained with the aid of a Wiener process. It is enough to put

\[ X_t = \min\{v : W_v \leq t\}, \]

see Ito and McKean (1965).

**Theorem (Lukacs (1967))**

Let \( (X_t)_{t \geq 0} \) be a homogeneous continuous process with independent increments. Suppose further that \( X_t \) is non-degenerate, the distribution of the increments is symmetric and that \( X(0) = 0 \). The process \( (X_t)_{t \geq 0} \) is S\( \alpha \)S iff there exists a function \( t_y \) such that

(i) \( t_y > 0 \) for all \( y > 0 \)

(ii) the stochastic integral \( \int_0^y (y - t) dX_t \) has the same distribution as \( X_y \) for all \( y > 0 \). The integral is understood in the sense of \( y \)-convergence in probability.

The assumption of symmetry is not essential here and a similar result holds in a more general setting. The next characterization of the \( \alpha \)-stable motion holds for \( \alpha > 1 \) and uses a regression property.

**Theorem (Praksa-Rao (1968))**

Let \( (X_t)_{0 < t < 1} \) be a homogeneous continuous process with independent increments. Suppose further that

(i) \( X(0) = 0 \) and \( \mathbb{E} X_t = 0 \) for all \( t \),

(ii) the increments of \( X_t \) have symmetric non-degenerate distributions.

Let \( Y_\lambda = \int_0^t u^\lambda dX_u, \lambda > 0 \), the process \( X_t \) is a \( \text{S}^\alpha \text{S} \) process iff for some
positive $\mu$ and $\lambda$, $\mu \neq \lambda$, the relation

$$\mathbb{E}(X_{\lambda} | Y_{\mu}) = \beta Y_{\mu}$$

holds almost everywhere.

Here $\alpha$ is determined in terms of $\lambda, \mu$ and $\beta$.

B. $\alpha$-Sub-Gaussian processes.

A process $X$ is called $\alpha$-sub-Gaussian if its finite dimensional characteristic functions have the form

$$\mathbb{E}\exp\left(i \sum_{n=1}^{N} a_n X_n\right) = \exp\left(-\frac{1}{2} \sum_{m,n=1}^{N} a_m a_n R(t_m, t_n)\right)^{\alpha/2},$$

where $R(t,s)$ is a positive definite function. It is well known that

$$X_t = A^{1/2} Y_t, \quad -\infty < t < \infty,$$

where $A$ is a positive $\alpha/2$ stable r.v. independent of the Gaussian process $Y_t$ which mean zero and covariance function $R$.

This very simple class of stable processes has been considered by many authors: Bretagnolle et al (1966), Miller (1978), Cambanis and Miller (1981), Hardin (1982) and so on.

It has some exceptional properties. While linear spaces of Gaussian r.v.'s are rich in (nondegenerate) independent elements, this is no longer true in the sub-Gaussian case.

EXAMPLE. (Cambanis, Soltani 1983).

Let $X_1, X_2$ be nondegenerate, jointly $\alpha$-sub-Gaussian. Then

$$\mathbb{E}\exp(i t_1 X_1 + t_2 X_2) = \exp\left(-\frac{1}{2}(r_{11} r_{12} t_1 t_2 + t_2^2 r_{22})\right)^{\alpha/2}.$$

If $X_1$ and $X_2$ would be independent, then also

$$\mathbb{E}\exp(i t_1 X_1 + t_2 X_2) = \mathbb{E}\exp i t_1 X_1 \cdot \mathbb{E}\exp i t_2 X_2 =$$

$$\exp\left[-\left(\frac{1}{2} d t_1 r_{11}\right)^{\alpha/2} - \left(\frac{1}{2} d t_2 r_{22}\right)^{\alpha/2}\right].$$

Since these two expressions are equal for all $t_1, t_2$ we obtain (putting $a = r_{12}(r_{11} r_{22})^{-1/2} \in [-1,1]$)
\[ x^2 + 2ax + 1 = (|x|^a + 1)^{2/a} \] for all \( x \),

but this cannot be true for any \( a \in [-1,1] \).

C. Harmonizable processes

A complex valued SoS process \( X_t \) on a LCA group \( T \) is called harmonizable if

\[
X_t = \int_A <t,y> W(dy), \quad t \in T
\]

where \( W \) is a complex independently scattered SoS measure on the Borel \( \sigma \)-algebra of the dual group \( \hat{A} \) with the finite control measure \( F \) i.e., \((\hat{A}, B_{\hat{A}}, F)\). Such processes are considered by Weron (1983) and earlier for the case \( T = \mathbb{Z} \) by Hosoya (1983) and for \( T = \mathbb{R} \) by Cambanis and Soltani (1982). Cf. also Marcus and Pisier (1982).

Harmonizable processes are covariation stationary in the sense that its covariation function \([X_t, X_{t+s}]_\alpha\) doesn't depend on \( s \) and can be represented by

\[
[X_t, X_s]_\alpha = \int_A <t-s, y> d\mathbb{P}(\gamma), \quad 1 < \alpha < 2
\]

**EXAMPLE**

Let \( X_t = \int_{-\infty}^{\infty} e^{it\lambda} W(dx) \) be a harmonizable SoS process \( 1 < \alpha < 2 \), on the real line. If it is regular i.e., \( \cap L(X:t) = \{0\} \) then the time domain \( L(X: + \infty) \) contains no independent r.v.'s. Indeed, by Beurling's theorem for each \( t \in \mathbb{R} \), \( L(X: t) = \{ \int_{-\infty}^{\infty} f dZ : f \in L^\alpha(F), \hat{f} = 0 \text{ on } (t, + \infty) \} \), where \( Z \) is a \( \alpha \)-stable motion, and \( \hat{f} \) the Fourier transform of \( f \), (cf. Cambanis and Soltani (1983) th. 3.1). Since these functions are boundary values of analytic functions, they cannot vanish on positive Lebesgue measure sets unless they are identically zero. Thus \( L(X: t) \) contains no nodegenerate independent r.v.'s as \( \int_{-\infty}^{\infty} f dZ \) and \( \int_{-\infty}^{\infty} g dZ \) are independent iff \( |f_1| \cdot |f_2| = 0 \text{ a.e. [Leb.]} \).

**Theorem.** (Weron 1983)

Let \( (X_t)_{t \in T} \) be a harmonizable SoS process on a LCA group \( T \) and \( 1 < \alpha < 2 \). Then,

(A) There exists a preserving semi-inner product correspondence (an isometric isomorphism \( \Gamma \)) between the time domain \( L(X:T) \) of the process and its spectral domain \( L^\alpha(F) \) given by

\[
\Gamma p(\gamma) = \int_A p(\gamma) W(dy), \quad p(\cdot) \in L^\alpha(F).
\]
If a harmonizable SaS process $Y_t$ is left stationarily related to $X_t$ (i.e., there exists a finite measure $F_{YX}$ such that $[Y_t, Y_s]_a = \int_A \langle t - s, \gamma \rangle F_{YX}(d\gamma)$) then the following conditions are equivalent

(i) $Y$ is subordinate to $X$ i.e., $L(Y ; T) \subseteq L(X ; T)$
(ii) there exists a function $p(\gamma) \in L^a(F_X)$ such that $Y_t = \int_T \langle t, \gamma \rangle p(\gamma) \mathcal{W}_X(d\gamma)$, $t \in T$
(iii) there exists a function $p(\gamma) \in L^a(F_X)$ such that $F_Y(\Delta) = \int_\Delta |p(\gamma)|^a F_X(d\gamma)$ and $F_{YX}(\Delta) = \int_\Delta p(\gamma) F_X(d\gamma)$.

In contrast with the case $\alpha = 2$ the result in part B is nonsymmetric and we need to consider left or right stationarily related processes. For the right related processes the second condition in (iii) reads as follows

$$F_{YX}(\Delta) = \int_\Delta p(\gamma) \langle \alpha - 1 \rangle F_X(d\gamma).$$

**EXAMPLE**

Let $(\Theta_k,k \in \mathbb{Z})$ be an i.i.d. collection of complex valued $\alpha$-stable r.v.'s and let $(a_k)$ be a sequence of real or complex coefficients satisfying $\sum |a_k|^\alpha < \infty$. If we put

$$X_t = \sum_{k=-\infty}^{\infty} a_k \Theta_k e^{i2\pi k t}, \quad t \in [0,1] = T$$

then the random Fourier series $X_t$ is a harmonizable SaS process, $0 < \alpha < 2$, with the control measure $F = \sum_k |a_k|^\alpha \delta_k$, where $\delta_k$ is the unit point mass at $k \in \mathbb{Z}$. Of course, this example can be extended to any compact Abelian group $T$ by replacing $\mathbb{Z}$ by the discrete dual group $\hat{T}$ and $e^{i2\pi k t}$ by characters $\langle t, \gamma \rangle$.

**D. Stationary processes.**

A stochastic process $(X_t)_{t \in T}$ is called stationary if $(T, +)$ is a group and the random vectors $(X_{t_1}, \ldots, X_{t_n})$ and $(X_{t_1 + s}, \ldots, X_{t_n + s})$ are identically distributed for each choice of $s, t_1, \ldots, t_n \in T$.

Stationary SaS processes form a richer class of processes than the stationary Gaussian processes. For instance, while all continuous in $L^2$ stationary Gaussian processes have a harmonic spectral representation on any LCA group $T$, this is no longer valid in the stable case.
EXAMPLE

A moving average SaS process is defined by

\[ X_t = \int_{-\infty}^{\infty} f(t - \lambda) dZ(\lambda) \quad \Rightarrow t < \infty, \]

where \( Z \) is a real \( \alpha \)-stable motion and \( f \in L^\alpha(d\lambda) \). It is clearly stationary. In sharp contrast with the Gaussian case the class of SaS moving averages is disjoint from the class of regular harmonizable SaS processes. (cf. Cambanis and Soltani (1982) Th. 3.3). Let us recall that in the Gaussian case these two classes coincide.

EXAMPLE

A harmonizable SaS process \( X_t = \int_T \langle t, \gamma \rangle W(d\gamma) \) is stationary iff the random measure \( W \) is rotationally invariant, i.e., the distribution of the process \( \{ e^{i\phi} W(B), B \in B \} \) does not depend on \( \phi \). This is the so called Maruyama-Urbanik theorem for infinitely divisible processes. See Maruyama (1970), Urbanik (1968) and Cambanis (1983). Again unlike the Gaussian case \( \alpha = 2 \), where all continuous in probability stationary Gaussian processes are harmonizable, there are plenty of continuous in probability stationary SaS processes which do not have a harmonic representation, such as moving averages (which we knew from the previous example) and \( \alpha \)-sub-Gaussian processes. Observe that an \( \alpha \)-sub-Gaussian process \( X_t = A^{1/2} \gamma_t \) is stationary iff the Gaussian process \( \gamma_t \) is stationary. It can be represented as

\[ X_t = \int_T \langle t, \gamma \rangle (A^{1/2} \cdot G(d\gamma)) \], \ t \in T \]

where \( G \) is a Gaussian independently scattered measure. Since \( A^{1/2} \) is \( \alpha/2 \) stable and independent of \( G(\cdot) \) the mixture \( A^{1/2} G(\cdot) \) is a SaS measure, see §1, but it is no longer independently scattered.

Below we present the description of SaS stationary processes developed by Hardin (1983).

Theorem (Hardin 1982)

A non-Gaussian SaS process \( (X_t)_{t \in T}, 0 < \alpha < 2 \), where \( T \) is a group, is stationary iff it has the following representation in law

\[ \{X_t, t \in T\} = \{ \int \phi(U_t) \lambda(\lambda) dZ(\lambda), t \in T \} \]

where \((E, \Sigma, \mu)\) is a measurable space, \( \phi \in L^\alpha(E, \Sigma, \mu) \), \( U_t \) is a group of isometries on \( L^\alpha \) and \( Z \) is the canonical independently scattered SaS measure on \((E, \Sigma, \mu)\). When \( X_t \) is continuous in probability, \((E, \Sigma, \mu)\) can be chosen to
be the unit interval (or the real line $\mathbb{R}^1$ or $\mathbb{R}^k$) with Lebesgue measure.

EXAMPLE

While every $L^2$-stationary Gaussian process is stationary, this is not the case for harmonizable $\mathbb{S}\mathbb{S}$ processes as we have observed already. Consider a harmonizable process of the form $X_t = \int e^{i\lambda t}W(d\lambda)$ with the control measure $F$. We know that $X_t$ is stationary iff $W$ is rotationally invariant. Assume that $W$ is not rotationally invariant.

By Cambanis (1983) for any finite measure $F$ there exists a rotationally invariant $\mathbb{S}\mathbb{S}$ process $Z_t$ with independent increments whose control measure is exactly $F$. Define a new harmonizable $\mathbb{S}\mathbb{S}$ process by $Y_t = \int e^{i\lambda t}dZ(\lambda)$. It is stationary with the same control measure as $X_t$. Consequently, there exists an isomorphism between $L(X; \mathbb{R}) \rightarrow L(Y; \mathbb{R})$. Thus for each non-stationary harmonizable $\mathbb{S}\mathbb{S}$ process $X_t$ there exists a stationary harmonizable $\mathbb{S}\mathbb{S}$ process $Y_t$ which is isomorphic to $X_t$, i.e., $X_t = VY_t$, where $V: L(Y; \mathbb{R}) \rightarrow L(X; \mathbb{R})$ is an isometric isomorphism. This fact was observed first by Pourahmadi (1983).

Finally, let's consider a class of doubly stationary $\mathbb{S}\mathbb{S}$ processes introduced in Cambanis, Hardin and Weron (1984). For this let $(E, \Sigma, \mu)$ be an arbitrary measure space and let $\{f_t, t \in T\}$ be a collection of measurable functions on $E$ indexed by a parameter group $T$. Call $\{f_t\}$ stationary if $\mu(\{f_t+s, \ldots, f_t+s\} \in B)$ is independent of $s \in T$ whenever $n, t_j \in T$ and a Borel set $B$ of $\mathbb{R}^n$ are fixed. A $\mathbb{S}\mathbb{S}$ process $\{X_t; t \in T\}$ is called doubly stationary if it has the same distribution as some process $\{X_t; t \in T\}$ defined by $X_t = \int_E f_t(\lambda)dZ(\lambda)$, where $\{f_t\} \subset L^0(E, \Sigma, \mu)$ is a stationary family and $Z$ is the canonical independently scattered $\mathbb{S}\mathbb{S}$ measure on $(E, \Sigma, \mu)$. It is easy to see by checking characteristic functions that any doubly stationary $\mathbb{S}\mathbb{S}$ process is also stationary. It is known also that the converse does not hold, however all mean-zero stationary Gaussian processes, all stationary $\sigma$-sub-Gaussian processes and all $\mathbb{S}\mathbb{S}$ moving averages are doubly stationary.

§3. Spectral Representation

Let $(X_t)_{t \in \mathbb{R}}$ be a real $\mathbb{S}\mathbb{S}$ process. Since each r.v. $Y$ in the time domain of the process $L(X) = \text{sp}(X_t : t \in \mathbb{R})$ has a $\mathbb{S}\mathbb{S}$ distribution, there is a number $c > 0$ such that the characteristic function for $Y$ is given by

$$E \exp(itY) = \exp(-c|t|^\alpha).$$
It is known that letting $\|Y\|_n = c^{1/\alpha}$ defines a quasi-norm which metrizes convergence in probability. For $1 < \alpha < 2$ the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\alpha$. It is also known, that each finite-dimensional subspace of $L(X)$ imbeds linearly and isometrically into $L^\alpha[0,1]$. This implies by Bretagnolle at al. (1966), for the case $p > 1$, Schreiber (1972) for the case $p < 1$ that there exists a measure space $(E,\mu)$ and a linear isometric imbedding of $L(X)$ into $L^\alpha(E,\mu)$. See also Schilder (1970), Kanter (1973) and Kuelbs (1973).

Consequently, we may represent the characteristic function of a complex S\&S process as

\[
\mathbb{E} \exp \left( i t_1 \Re Y + t_2 \Im Y \right) = \exp \left( -\left| t_1 \right|^{\alpha} \sum_{j} \sum_{j'} \left| \int_{E} f_{j} \, d\mu \right|^{\alpha} \right)
\]

where $Y = \sum_{j} X_{t_j}$, $\{f_{t_j}; t \in \mathbb{R}\} \subset E^\alpha$. Conversely, The Kolmogorov theorem implies that for any choice of the $f_{t_j}$'s in an $L^\alpha$-space (*) defines a stable process $X_t$.

The map $t + f_t$ is called a spectral representation for the process $X_t$. In the case $L(X)$ is separable one can choose $(E,\mu)$ to be $[0,1]$ with Lebesgue measure. Then (*) can be translated to

\[
X_t = \frac{1}{\mu} \int_0^1 f_t(s) \, dZ_t, \quad t \in \mathbb{R},
\]

where $Z_t$ is an $\alpha$-stable motion. The following concept of minimal spectral representation introduced by Hardin (1982) "puts the dots on the i's" in the long study.

Let $F = \mathbb{sp} \{f_t; t \in \mathbb{R}\}$ in $L^\alpha(E,\mu)$ and $\rho(F) = \sigma(f/g; f, g \in F)$. The map $t + f_t \in L^\alpha(E,\Sigma,\mu)$ is called a minimal representation if

(i) there is no set $B \in \Sigma$ with $\mu(B) > 0$ such that $f_t = 0$ a.e. $\mu$ on $B$ for all $t \in \mathbb{R}$

(ii) $\rho(F) = \Xi$.

Observe that if $L(X)$ is separable then $L^\alpha(E,\mu)$ is isometric to either $L^\alpha[0,1]$, $L^\alpha[0,1] + \mathbb{R}^n$ or $L^\alpha[0,1] + \mathbb{R}^n$ according to whether $\rho(F)$ has no atoms, $n$-atoms, or infinitely many atoms. (cf. Lacey (1974) p. 128).

**Theorem** (Hardin 1982)

(i) Every complex S\&S process has a minimal representation.

(ii) Each two minimal representations for a given non-Gaussian S\&S process are isometrically equivalent.
EXAMPLE

a) Each $\alpha$-stable motion $Z$ has minimal representation $t + l(0,t)$.

b) Each mean zero Gaussian process $X_t$ on $(\Omega, \Sigma, P)$ can be represented by $Y_t = \int_0^T X_t(\omega)dZ(\omega)$, where $Z$ is the canonical independently scattered Gaussian measure on $(\Omega, \Sigma, P)$. Since the characteristic functions of $X_t$ and $Y_t$ coincide these processes are stochastically equivalent. To see this observe

$$i\Sigma u X_t - 1/2\var(X_i X_j) = e^{1/2\Sigma u u \text{Cov}(X_i, X_j)}$$

The spectral representation is the mapping $t + X_t \in L^2(\Omega, P)$, which of course is not very useful.

c) Each continuous in probability $\alpha$-sub-Gaussian process $X_t = A^{1/2}Y_t$ on $(\Omega, \Sigma, P)$ has a spectral representation $t + cY_t$ on $(\Omega, \Sigma, P)$. But it is not minimal, (see Hardin (1982) p. 393). Here $c$ is a constant depending only on $\alpha$.

d) A continuous in probability non-Gaussian SGS process on a group $T$ is stationary iff it has a minimal representation of the form $t + P_t\phi$, where $\phi$ is a fixed function in $L^2[0,1]$ and $\{P_t : t \in T\}$ is a group of isometries on $L^2[0,1]$, (Hardin (1982), cf. §2.D).

§4. Series and Integral Representation

Any Gaussian process with a.s. sample paths continuous (or equivalently a Gaussian r.v. with values in a separable Banach space) can be represented as an a.s. convergent series

$$X_t = \sum_{n=1}^{\infty} \gamma_n a_n(t)$$

Here the $\gamma_n$ are i.i.d. standard Gaussian r.v.'s and $a_n(t) \in C[0,1]$ (or the Banach space). cf. Kallianpur and Jain (1970) or LePage (1972). It is known that there are no such representations for SGS r.v.'s, see §10, and consequently for SGS processes.

We will say that a SGS process $\{X_t, t \in T\}$ has a series expansion if $(X_t')_{t \in T}$ is distributed as $(X_t)_{t \in T}$ where

$$X_t' = \sum_{k=1}^{\infty} a_k(t)\theta_k,$$
where the \( \Theta_k \) are i.i.d. SGS r.v.'s and the convergence is a.s. for each fixed \( t \).

**Theorem (Cambanis, Hardin and Weron (1983)).**

Let \( (X_t)_{t \in T} \) be a SGS process with separable time domain \( L(X) \). Then \( (X_t)_{t \in T} \) can be represented as

\[
X_t = X^1_t + X^2_t,
\]

where \( X^1_t \) and \( X^2_t \) are independent SGS processes, \( X^1_t \) has a series expansion and \( X^2_t \) has no non-trivial series expansion. This decomposition for discrete and diffuse components is unique up to distribution. Moreover, the process \( X_t \) has only the discrete component \( X^1_t \) iff it has a minimal spectral representation on \( L^q \) or \( L^q_n \).

**EXAMPLE**

It follows from a discussion in §2 that \( \alpha \)-sub-Gaussian processes and regular harmonizable SGS processes have no discrete components since their linear spaces contain no independent r.v.'s at all.


Let \( \{\tau_j, j \geq 1\} \) be i.i.d. taking values in a measurable space. \( \{\tau_j, j \geq 1\} \) are arrival times of a Poisson process with unit rate, \( \{\varepsilon_j, j \geq 1\} \) are i.i.d. with \( P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = 1/2 \) and \( \{X_j, j \geq 1\} \) are i.i.d. with \( \mathbb{E}|X_1|^{\alpha} < \infty \).

**Theorem (LePage (1981))**

Suppose \( \tau_j, \varepsilon_j, X_j \) and \( \Gamma_j \) are mutually independent and define for each measurable \( A \)

\[
Z(A) = \sum_{j=1}^{\infty} \frac{1}{(\tau_{j} \epsilon A )^{X_j} \varepsilon_j^{1-1/\alpha}}.
\]

The above series is a.s. convergent for each \( A \), is jointly SGS for finitely many \( A \)'s at a time, and \( Z(A_1), \ldots, Z(A_n) \) for \( n > 1 \) are mutually independent if \( A_1, \ldots, A_n \) are mutually disjoint.

**EXAMPLE**

By the above theorem we can express the spectral representation of any SGS process as follows
\[ X_t = \int_{-\infty}^{\infty} f_t(\lambda) Z(d\lambda) = \sum_{j=1}^{\infty} f_t(\tau_j) X_j e_j^{-1/\alpha}, \]

where \( f_t \in L^\alpha(\mu), \mu \) is the distribution of \( \tau_j \) and

\[ Z(\lambda) = \sum_{j=1}^{\infty} 1(\tau_j < \lambda) X_j e_j^{-1/\alpha}. \]

This LePage's representation provides a useful interpretation of a SaS process. Namely, by choosing \( \epsilon_j = Z_j / (\|Z_j\|^{1/\alpha}), j \geq 1 \) where \( \{Z_j\} \) are i.i.d. standard Gaussian variables, conditioned on the sequences \( \{\tau_j\}, \{X_j\} \) and \( \{\Gamma_j\} \), the process \( X_t \) is a.s. Gaussian.

Multiplicity representations for Gaussian processes (or for general \( L^2 \)-process) were discovered independently by Cramer (1960) and Hida (1960). Those explicit representation of a Gaussian process through the Brownian motion depend heavily on Hilbert space methods. For aSaS processes the time domain can be consider as an \( L^\alpha \) space, however there is no analogue of Hellenier-Hahn multiplicity theory here. It turns out that in sharp contrast with the Gaussian case there are 4 different types of multiplicity representations for aSaS processes. These types are where the innovation processes are independent (or James orthogonal) aSaS processes with independent (or James orthogonal) increments, respectively (Cambanis, Hardin and Weron (1983)).

We present here two of them, namely Hida type and Cramer type. In general Hida type \( \Rightarrow \) mixed types \( \Rightarrow \) Cramer type, but not conversely.

**Example**

Let \( Z^1, Z^2 \) be independent \( \alpha \)-stable motions on \([0, +\infty)\), let \( X_t = Z^1_t \) if \( t \) is rational and \( X_t = Z^2_t \) if \( t \) is irrational. It is easy to see that \( (X_t)_{t \geq 0} \) is a SaS process with the following Hida-type multiplicity representation

\[ X_t = \sum_{k=1}^{2} \int_{0}^{t} g_k(t, u) dZ^k(u), \]

where \( g_k(t, u) = 1_{E_k}(u), E_1 \)-rational and \( E_2 \)-irrational numbers.

**Example**

Let \( X_t = A^{1/2} Y_t \) be a \( \alpha \)-sub-Gaussian process. Assume that \( X_t \) is a real, left continuous and regular, then so is the Gaussian process \( Y_t \). By Cramer-Hida theorem \( Y_t \) has the multiplicity representation

\[ Y_t = \sum_{k=1}^{N} \int_{0}^{t} f_k(t, u) dB^k(u), \]
where \( B^k(u) \) are independent Brownian motions. Consequently multiplying by \( A^{1/2} \), where it is \( \alpha/2 \) stable and independent of \( Y_t \) one gets the following Cramer-type multiplicity representation

\[
X_t = \sum_{k=1}^{\infty} f_k(t,u)Z^k(u),
\]

where \( Z^k(u) = A^{1/2}B^k(u) \) are \( \text{SaS} \) process, but they are not independent. Instead \( Z^k \) are James orthogonal and with James orthogonal increments.

In order to characterize Hida type multiplicity the following concept is needed. A \( \text{SaS} \) process admits the independent projection property (I.P.P.) if

(i) for each \( t \in \mathbb{R} \) there exists an independent projection \( Q_t \) on \( L(X : t) \)
(ii) for each r.v. \( V \in L(X) \) there exists an independent projection \( Q \) onto \( N_V = \text{sp}(Q_tV, t \in \mathbb{R}) \).

Observe that there are \( \text{SaS} \) processes which do not admit (I.P.P.); \( \alpha \)-sub-Gaussian or regular harmonizable \( \text{SaS} \) processes. cf §2.

**Theorem** (Cambanis, Hardin, and Weron (1983))

A. Let \( (X_t)_{t \in \mathbb{R}} \) be a \( \text{SaS} \) process \( 1 < \alpha < 2 \), such that

(i) \( X_t \) is left-continuous in \( \| \cdot \|_{\alpha} \)-norm
(ii) \( X_t \) is regular i.e., \( \cap L(X : t) = (0) \)
(iii) \( X_t \) admits I.P.P.

Then \( X_t \) has a Hida type multiplicity representation

\[
X_t = \sum_{n=1}^{\infty} g_n(t,u)Y^n(u),
\]

where \( Y^n(u) \) are mutually independent \( \text{SaS} \) processes with independent increments, the spectral functions \( g_n(t) = \text{sp}(Q^nY^n)_{\alpha} \) satisfy the relation

\[
G_1 > G_2 > \ldots > G_N,
\]

(i.e., \( G_N \) is absolutely continuous w.r.t. any \( G_n \)) and \( g_n(t,u) \in L^\alpha(G_n(\cdot)) \) for each \( t \in \mathbb{R} \).

B. In any two such representations the multiplicity \( N \) will have the same value, and the \( G_n \)'s will be pairwise equivalent.

C. Any \( \text{SaS} \) process with the Hida type multiplicity representation admits I.P.P.
The concept of local nondeterminism introduced for Gaussian processes by Berman (1973) has been extended to SαS process by Nolan (1982) in the following way.

A SαS process \((X_t)_{t \in \mathbb{T}}\) is **locally nondeterministic** on \(\mathbb{T}\) if its spectral representation \(t + f_t \in L^\alpha\) satisfies

(a) \(\|f_t\| > 0\) for all \(t \in \mathbb{T}\)

(b) \(\|f_t - f_s\| > 0\) for all \(t, s \in \mathbb{T}\) with \(|t - s|\) sufficiently small

(c) For all \(m > 2\) and any ordered \(t_1 < t_2 < \ldots < t_m\) in \(\mathbb{T}\)

\[
\lim_{\epsilon \to 0} \inf_{t_m - t_{m-1} < \epsilon} \frac{\|f_{t_m} - f_{t_{m-1}}\|}{\|f_{t_1} - \ldots - f_{t_{m-1}}\|} > 0.
\]

The above expression is a relative prediction error. The ratio is always between 0 and 1. If it is positive as \(|t_m - t_1| \downarrow 0\), this can be interpreted as saying that knowledge of \(X_t\) at the closest time point \(t_{m-1}\) gives roughly the same order of information about \(X_t\) as knowledge of \(X_t\) at the preceding times \(t_1, \ldots, t_{m-1}\). The term in (c) above is the analogue of the conditional variance used in Berman's original definition.

Here we present a sufficient condition for local nondeterminism of SαS processes with the Hida type multiplicity one. Let us mention that the concept of local nondeterminism is used in proving existence and regularity of local times.

**Theorem (Nolan (1982)).**

Let \((X_t)_{t \in \mathbb{R}}\) be a SαS process with the Hida type multiplicity one i.e.,

\(X_t = \int_{\mathbb{R}} g(t, u) dZ(u)\). If \(\|g(t, \cdot)\| > 0\), \(\|g(t, \cdot) - g(s, \cdot)\| > 0\) for all distinct \(t, s \in \mathbb{R}\) and

\[
\lim_{\epsilon \to 0} \inf_{t \in \mathbb{R}} \frac{\int_{s}^{t} |g(t, u)|^\alpha du(u)}{\int_{s}^{t} |g(t, u) - g(s, u)|^\alpha du(u)} > 0,
\]

then the process is locally nondeterministic.

**EXAMPLE.**

Fix \(0 < \alpha < 2\) and any \(g(\lambda) \in L^\alpha(\mathbb{R}, d\lambda)\) with \(\text{supp } g \subset (-\mathbb{R}, 0]\). If \(W\) is \(\alpha\)-stable motion then the process
is stationary. By the above result if

$$\lim_{n \to 0} \frac{1}{n} \int_{-h}^{0} |g(\lambda)| d\lambda > 0$$

then the process is locally nondeterministic. Taking $g = 1[-t,0]$ for some $t > 0$ we have that above limit is positive and consequently $W_t$ is locally nondeterministic.

For any $0 < \alpha < 2$ and any $\beta > -1/\alpha \ g(\lambda) = e^{\lambda |\lambda|^{\beta}} 1_{(-\infty,0)}(\lambda)$ is in $L^2(\mathbb{R}, d\lambda)$. It is easy to see that the corresponding $\text{SaS}$ process is locally nondeterministic whenever $-1/\alpha < \beta < 1 - 1/\alpha$. When $\alpha = 2$, this is Berman's example.

§5. Ergodic Properties

We now turn our attention to laws of large numbers. If $X_t = \int \exp{i\lambda \cdot Z(d\lambda)}$ is a $\text{SaS}$ harmonizable process then it is easily seen that

$$\frac{1}{T} \int_0^T X_t dt = \int \frac{1}{T} \int \exp{i\lambda \cdot Z(d\lambda)} \to \int \mathbb{1}_{\{0\}}(\lambda) d\lambda = Z(0).$$

in probability. This can be also extended to the processes on a second countable LCA group if the intervals $[0,T]$ are replaced by a regular sequence of compact subsets, cf. Weron (1983). Thus $X$ satisfies a WLLN iff $Z(0) = 0$.

Similarly as in the Gaussian case it tells that the time average of the process is a consistent estimate of the mean iff the control measure $F(\cdot)$ is continuous at $0$, since we have $F(0) = \text{IZ}(0) 1_{\mathbb{R}} = 0$. While this is useful to know, it is not general enough from the statistical point of view, since it only tells us something about a particular parameter of the process, the mean, and a particular estimate of it. To probe deeper into the consistency question, one must consider more general parameters. The question of consistent estimation leads us to study stationary processes and their ergodic properties. It is well known that ergodicity is equivalent to metric transitivity. Recall that a stationary process is called metrically transitive if all shift invariant events have probability zero or one.

By a result of Maruyama (1949) and Grenander (1950) a stationary Gaussian process is metrically transitive iff its spectral measure has no atoms. For $\text{SaS}$ processes we have the following characterization.
Theorem (Cambanis, Hardin, Weron (1984)). A real stationary SA S process $X_t$ with $0 < \alpha < 2$ and spectral representation $\int V_t \phi(\lambda) dZ(\lambda)$ is metrically transitive iff for each $h \in \text{sp}\{V_t \phi, \ t \in \mathbb{R}\}$ $L^\alpha(\mu)$

(i) \[ \lim_{T \to \infty} \frac{1}{T} \int_0^T V_t h - h t^\alpha dt = 2 t h t^\alpha \]

and

(ii) \[ \lim_{T \to \infty} \frac{1}{T} \int_0^T V_t h - h t^{2\alpha} dt = 4 t h t^{2\alpha}. \]

When a stationary SA S process $X_t$ is metrically transitive one can estimate its covariation function, namely for $1 < \alpha < 2$ and $0 < p < \alpha$

\[ \frac{1}{T} \int_0^T X_t (X_{t+\tau})^{(p-1)} dt \xrightarrow{T \to \infty} E[X_0 (X_\tau)^{(p-1)}] = C(p, \alpha) \frac{\text{Cov}(X_\tau, X_{\tau})}{1 X_\tau^{(p-1)}}, \]

Corollary (Cambanis, Hardin, Weron (1984)).

a) Every moving average SA S process is mixing and consequently is metrically transitive.

b) A $\alpha$-sub-Gaussian process is never metrically transitive.

c) A stationary harmonizable SA S process is never metrically transitive when $0 < \alpha < 2$.

d) A double stationary SA S process is metrically transitive if

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu(A \cap T_t B) dt = 0 \text{ for all sets } A, B \in E \text{ of finite } \mu\text{-measure}. \]

Thus an analogue of the Maruyama-Grenander theorem doesn't hold for stable processes which follows from part c.

The result c) was first obtained in LePage (1980) by using his series representations. A description of ergodic decomposition for $\alpha$-sub-Gaussian and also for stationary harmonizable SA S processes is given in Cambanis, Hardin, Weron (1984).

If $1 < \alpha$ then $E|X_t| < \infty$ and by the classical Birkhoff ergodic theorem a SLLN holds for stationary SA S processes

\[ \frac{1}{T} \int_0^T X_t dt \xrightarrow{a.s.} E(X_1 | J), \]

where $J$ is the $\sigma$-field of invariant sets. Consequently, stationary SA S
process satisfies a SLLN iff it satisfies a WLLN.

A much more complicated question is to find a SLLN for non-stationary processes. However, for harmonizable \( \text{SaS} \) processes we can adapt Gaposhkin's (1977) approach from the \( L^2 \)-case.

**Theorem (Cambanis, Hardin, Weron (1984)).**

If \( (X_t)_{t \in \mathbb{R}} \) is a complex harmonizable \( \text{SaS} \) process then

\[
\frac{1}{T} \int_0^T X_t dt \rightarrow Z(0) \quad \text{a.s.}
\]

§6. Path Properties

Cambanis and Rajput (1973) have shown, by using a general zero-one law of Kallianpur that most regularity properties of sample paths of real-valued Gaussian processes hold with probability zero or one. Cambanis (1973) has obtained necessary and sufficient conditions for almost sure absolute continuity of sample paths and Dudley (1973) and Fernique (1975) have obtained necessary and sufficient conditions for almost sure continuity of sample paths of real valued stationary Gaussian processes.

That these regularity properties of sample paths also hold for \( \text{SaS} \) processes has been pointed out by Cambanis and Miller (1980) and by Marcus and Pisier (1982), respectively. The first result will be discusses in §9 in connection with a general zero-one law. Here we are going to consider extension of Dudley-Fernique theorem.

Let \( X_t \) be a stationary harmonizable \( \text{SaS} \) process, \( 0 < \alpha < 2 \) on a LCA group \( T \) i.e., \( X_t = \int_A \langle t, \gamma \rangle \mathcal{W}(d\gamma) \). Marcus and Pisier (1982) have introduced a class of so called "strictly stationary processes" but it is easy to see that their class coincides with the class of stationary harmonizable \( \text{SaS} \) processes, since

\[
\mathbb{E} \exp\left[ i \sum_{j=1}^n a_j X(t_j) \right] = \exp\left(-\int_A \sum_{j=1}^n a_j \langle t_j, \gamma \rangle |^{\alpha} d\mathcal{F}(\gamma) \right).
\]

Associate with \( X_t \) a pseudo-metric \( d_X \) on \( T \) defined by

\[
d_X(s,t) = \left[ \int_A |\langle s, \gamma \rangle - \langle t, \gamma \rangle |^{\alpha} d\mathcal{F}(\gamma) \right]^{1/\alpha}.
\]

Let \( K \) be a fixed compact neighborhood of the unit element of zero, and let \( N(K,d_X; \varepsilon) \) denote the smallest number of open balls of radius \( \varepsilon \), in the pseudo-metric \( d_X \), which cover \( K \).

The following result extends the Dudley-Fernique characterization of the
a.s. continuity of sample paths of Gaussian stationary processes to \( \alpha \)-stable processes.

**Theorem (Marcus, Pisier (1982))**

Let \( 1 < \alpha < 2 \) and \( \beta \) be the conjugate of \( \alpha \). Let \( (X_t)_{t \in T} \) be a stationary harmonizable \( \mu \alpha \mu \) process on a LCA group \( T \). Then \( (X_t)_{t \in K} \) has a version with a.s. continuous sample paths iff

\[
J_\beta(d_x) = \int_0^\infty \left[ \log N(K,d_x : \varepsilon) \right]^{1/\beta} d\varepsilon < \infty.
\]

The case \( \alpha < 1 \) is trivial since in this case the fact that \( F \) is a finite measure insures that the process \( (X_t)_{t \in T} \) has a.s. continuous paths. For \( \alpha = 1 \) the problem is open. Erhard and Fernique (1981) have shown that in general Slepian's lemma cannot be extended from Gaussian to \( \alpha \)-stable processes, however if \( X_1 \) and \( X_2 \) are two stationary harmonizable \( \mu \alpha \mu \) processes such that

\[
\forall s,t \in K \quad d_{X_1}(s,t) < d_{X_2}(s,t)
\]

then the a.s. continuity of \( (X_2(t))_{t \in K} \) implies that of \( (X_1(t))_{t \in K} \). Moreover, the following comparison principle holds.

**Theorem (Marcus, Pisier (1982))**

Let \( T \) be a finite set of cardinality \( n \) and let \( (X_t)_{t \in T} \) and \( (Y_t)_{t \in T} \) be two stochastic processes such that \( (X_t)_{t \in T} \) is a \( \mu \alpha \mu \) process \( 0 < \alpha < 2 \) and \( (Y_t)_{t \in T} \) is a Gaussian process (i.e., \( \alpha = 2 \)). If for \( \forall s,t \in T \quad d_Y(s,t) < d_X(s,t) \), then for each \( r < p \) there exists a constant \( B(\alpha,r) \) such that

\[
(1) \quad \left[ \mathbb{E} \sup_{s,t \in T} |Y_s - Y_t|^r \right]^{1/r} < B(\alpha,r)(\log n)^{\alpha - \frac{1}{r}} - \frac{1}{2}\left[ \mathbb{E} \sup_{s,t \in T} |X_s - X_t|^r \right]^{1/r}
\]

In particular if \( 1 < \alpha < 2 \), then

\[
(2) \quad \mathbb{E} \sup_{t \in T} Y_t < B(\alpha,1)(\log n)^{\alpha - \frac{1}{2}} \mathbb{E} \sup_{t \in T} X_t.
\]

There exists an extensive literature on sample paths properties of the \( \alpha \)-stable motion, starting from Khintchine (1938). We refer the readers to the monograph notes of Mijnheer (1975), where generalized laws of the iterated logarithm for small and large times are presented in details. Here we include only LIL for large times.
Theorem (Breiman (1968))

Let \( \{X_t : 0 < t < \infty\} \) be a \( \alpha \)-stable motion with \( 0 < \alpha < 1 \), \( \alpha = 1 \) and \( \beta = 1 \). Let \( \phi \) be a positive, continuous and non-increasing function and take

\[
\psi(t) = \left\{ 2B(\alpha) \right\}^{1/2} \left\{ \phi(t) \right\} - \frac{\alpha}{2(1-\alpha)}
\]

where \( B(\alpha) = \frac{\alpha}{1-\alpha}(\cos(\pi\alpha/2)) \). Then

\[
P\left[\{w : \text{there exists some } t_0(w) > 0 \text{ such that } X(t,w) > t^{1/\alpha}\phi(t) \right. \text{ for all } t > t_0(w)\} = 0 \text{ or } 1
\]

according as the integral

\[
I(\psi) = \int_0^\infty \psi(t) t^{-1/2} \exp(-\frac{1}{2} \psi^2(t)) dt
\]

diverges or converges.

This is an complete analogue of Motoo's (1959) result for the Brownian motion which implies Khintchine's classical law of the iterated logarithm. Similarly here as a consequence of Breiman's result one gets.

Theorem (Fristedt (1964))

Under the assumptions of the previous theorem

\[
\lim_{t \to \infty} \frac{X_t}{t^{1/\alpha}(2 \log \log t)^{(1-\alpha)/\alpha}} = \left\{ 2B(\alpha) \right\}^{(1-\alpha)/\alpha} \text{ a.s.}
\]

Similar results, after necessary adjustments, hold also for the case \( \alpha = 1 \) and \( 1 < \alpha < 2 \) cf. Mijnheer (1975). The reason why we have to consider these four cases; \( \alpha = 2 \) the Gaussian case, \( 0 < \alpha < 1 \), \( \alpha = 1 \) the Cauchy case, and \( 1 < \alpha < 2 \), separately is that the tail of the distribution function of a completely asymmetric \( \alpha \)-stable r.v. differs in these cases.

The asymptotic behaviour of the first passage time and the sojourn time processes for a class of general processes with \( \alpha \)-stable motion components is given in Pruitt and Taylor (1969). The sample path growth at last exit time for \( \alpha \)-stable motion with \( 1 < \alpha < 2 \) is obtained in Monrad (1977).

The almost sure behaviour of the sample paths of the \( \mathbb{R}^2 \)-valued \( N \)-parameter \( \alpha \)-stable motions is investigated by Ehm (1981). In particular, it is shown that such a process has a.s. jointly continuous local times if \( N \alpha > d \). Also laws of the simple and of the iterated logarithm are established for the
supremum of the local time increments or sojourns times. These results give precise information on the minimum oscillation of the sample paths.

For more results on path properties of the $\alpha$-stable motion, the readers are referred to references in above mentioned papers. Let us note only that in all these works the term "$\alpha$-stable process" was used as the name for "$\alpha$-stable motion"! Also this improper terminology was established in Breiman's very popular book (Breiman (1968)) p. 316.

Finally, let us point out that the extremal behaviour of $\alpha$-stable moving averages, in particular ARMA-processes with $\alpha$-stable innovations is studied in Rootzen (1978) cf. also Leadbetter, Lindgren and Rootzen (1983).

§7. Linear Estimation
The linear theory of Gaussian processes, indeed of stochastic processes with finite second moments, has been fully developed. This includes linear estimation, and in particular prediction and filtering and the analysis and identification of linear systems with Gaussian inputs.

A basic difficulty in developing the linear theory of stable processes is due to the fact that while the linear span of a Gaussian process is a Hilbert space, the linear span of a stable process is a Banach space when $1 < \alpha < 2$ and only a metric space when $0 < \alpha < 1$.

For example, a regression involving jointly Gaussian random variables, is always linear, but this is not the case with all systems of jointly $\alpha$-stable variables (see Miller (1978), where he has obtained some necessary and sufficient conditions for the linearity of regressions). Kanter (1972) and Cambanis and Miller (1981) have also exhibited situations in which regressions are linear. A complete characterization of stable processes having linear spans in which all regressions are linear is as follows

Theorem (Hardin (1982))
A $\alpha$-stable process $1 < \alpha < 2$ has the linear regression property (i.e. $E(X_0 | X_1, \ldots, X_n) \in sp\{X_1, \ldots, X_n\}$ whenever $X_0, X_1, \ldots, X_n \in L(X; T)$) iff it is $\alpha$-sub-Gaussian.

Phrased another way, only for $\alpha$-sub-Gaussian processes measurable prediction (in the sense of conditional expectation) must agree with prediction in $L^\alpha$ sense ($L^\alpha$ metric projection).

Let $A$ be any proper non-empty subset of the parameter set $T$. The linear estimation problem arises when one wants to make linear predictions if exactly $X_t$ for $t \in T - A$ are known. That is to say, one is looking for a predictor
\( X_s \) of an unknown value \( X_s \) of the process based on a linear space of observations:

(i) \( X_s \in L(X; T - A) \), \( s \in A \)

(ii) \( \|X_s - X_s^a\|_\alpha = \inf_{Y} \|X_s - Y_s^a\| \),

where the infimum is taken over all \( Y \in L(X; T - A) \). It is known that \( X_s \) always exists for \( 1 < \alpha < 2 \) and it is obtained by a metric projection of \( X_s \) in the strictly convex Banach space \( L(X; T) \subset L^\alpha \). Thus it is the best approximation of \( X_s \) in \( L(X; T - A) \), see Singer (1970).

**EXAMPLE**

Let \( X_t \) be a \( \alpha \)-sub-Gaussian process. Then \( X_t = \Theta^{1/2} Y_t \) where a Gaussian process \( Y_t \) is independent of the positive \( \alpha/2 \)-stable r.v. \( \Theta \). We have

\[
\|X_s - X_s^a\|_\alpha = \inf_{Y \in L(X; T - A)} \|X_s - Y_s^a\|_\alpha
\]

\[
= \inf_{Y \in L(Y; T - A)} \left( \frac{1}{2} E[\|Y_s - Y_s\|^{2\alpha/2}] \right)
\]

and consequently \( X_s = \Theta^{1/2} Y_s \). Thus in this case the linear predictor is expressed in terms of the observables by the same "recipe" as in the corresponding Gaussian case. Moreover, the linear predictor is identical with the conditional expectation. This is easy to check, and as we already observed, it can happen only for \( \alpha \)-sub-Gaussian processes!

For other classes of stable processes linear problems turn out to be quite intractable in general. Recently some progress has been made for harmonizable \( SaS \) processes. The importance of this class is that their theory can be penetrated by Fourier analysis type arguments.

With the aim of carrying over \( L^2 \)-stationarity type arguments to the theory of \( SaS \) processes Hosoya (1982), Cambanis and Soltani (1982) have studied the extrapolation problem and Pourahmadi (1983) and Weron (1983) have studied the interpolation problem for the class of \( SaS \) harmonizable processes. Here we present some typical results:

Consider a \( SaS \) harmonizable sequence \( X_n \) i.e., of the form

\[
X_n = \int_0^{2\pi} e^{i\lambda n} dZ(\lambda) \quad n \in Z,
\]

where the control measure \( \mu(\cdot) = IZ(\cdot) \|_\alpha^a \) is finite. \( X_n \) is regular \( (\mu L(X; n) = 0) \) iff \( \mu \) is absolutely continuous and
\[ \int_0^{2\pi} \log \phi(\lambda) d\lambda = -\infty, \text{ where } \phi = d\mu/d\lambda. \] Let \( h \) be the outer factor of the density \( \phi \) i.e.,
\[ h(z) = \exp\left(\frac{1}{2\pi i} \int_0^{2\pi} \frac{2\pi i z}{\lambda - z} \log \phi(\lambda) \, d\lambda\right) \]
and
\[ h^V(m) = \frac{1}{2\pi} \int_0^{2\pi} h(\lambda) e^{-im\lambda} \, d\lambda \quad m = 0, \pm 1, \ldots \]

Denote
\[ \sum_{m=0}^{\infty} c(m) z^m = \left( h^V(0) + h^V(1) z + \ldots + h^V(N-1) z^{N-1}\right) \alpha/2 \]
for small \( |z| \). If \( P_N(z) = c(0) + c(1) + \ldots + c(N-1) z^{N-1} \) does not vanish in \( |z| < 1 \), then

**Theorem** (Cambanis and Soltani (1982)).

The linear predictor \( \hat{X}(n+N; n) \) of \( X(n+N) \) based on \( \{X_k, k < n\} \) has the form
\[ \hat{X}(n+N; n) = \int_0^{2\pi} e^{i\lambda(n+N)} (1 - \frac{\left[ P_N(e^{i\lambda})\right]^2}{h(\lambda)}) \, d\lambda \]
and the predictor error
\[ e_{\alpha}(N) = \frac{\hat{X}(n+N) - \hat{X}(n+N; n)}{\alpha} = \int_0^{2\pi} \frac{|P_N(e^{i\lambda})|^2}{h(\lambda)} \, d\lambda = \sum_{m=0}^{N-1} c^2(m). \]

The special case of one-step prediction (\( N = 1 \)) has been solved by Hosoya (1983). The problem is open when \( P_N(z) \) has roots in \( |z| < 1 \). Wide open also is the problem of finding the form of the linear extrapolation in the continuous parameter case. The problem of filtering a signal in the presence of additive noise using past observations is also open (even in the discrete case). The simpler problem of filtering from the past and future observations, which is useful when the parameter set is rather sparse than time has been considered by Cambanis and Miller (1981).

The interesting observation is made by Cambanis and Soltani (1982). Namely, the "Gaussian recipe" used in the stable case turns out to be as good as the optimal recipe, asymptotically as the prediction lag \( N \) tends to infinity. However, it should be noted that this is no longer true for interpolation problems where the formula for the prediction error depends on the index \( \alpha \).
Theorem (Weron 1983)

Let $T$ be a discrete Abelian group and $C$ a compact (hence finite) subset of $T$. Suppose the control measure $\mu$ of a harmonizable SåS process $(X_t)_{t \in T}$ is absolutely continuous w.r.t. the Haar measure $d\gamma$ of the dual group $T$ and such that $d\mu/d\gamma > 0$ a.s. $d\gamma$. Then

$$
\sup_{s \in T} \left| \frac{1}{s_{\alpha}} - \frac{1}{s_{\alpha}} + \frac{1}{s_{\alpha}} \right| = \max_{p \in P} \left[ \sup_{s \in T} \left| \frac{1}{s_{\alpha}} \right| \right]
$$

where $P = \{ p(\gamma) : p(\gamma) = \langle s, \gamma \rangle + \sum_{k \in C} a_k \langle t_k, \gamma \rangle \}$. If $p \in P$ and fulfills the condition (*) with $\sup_{s \in T} \left| \frac{1}{s_{\alpha}} \right| = \sigma$, then

$$
X_s = \int_T \left[ \langle s, \gamma \rangle - \frac{1}{s_{\alpha}} p(\gamma) \right] \frac{1}{s_{\alpha}} d\gamma + \sum_{k \in C} a_k \langle t_k, \gamma \rangle.
$$

For the case $T = Z$ the interpolation problem was also studied by Pourahmadi (1983). He has found algorithms for the linear interpolator and interpolation error of harmonizable SåS sequences. In contrast with the Gaussian case $\alpha = 2$ (cf. Salehi (1979)) this algorithm leads to a system of non-linear equations.

Consistent estimates of the spectral density function $\phi(\lambda)$ of real, fourth order, zero mean, $L^2$-continuous stationary processes have been studied extensively in the literature, cf. Brillinger (1975). Recently Masry and Cambanis (1982) have established weakly and strongly consistent nonparametric estimates, along with rates of convergence for the spectral density of certain stationary harmonizable SåS processes. Namely $0 < \alpha < 2, X_t = \int e^{it\lambda} dZ(\lambda), -\infty < t < \infty, (E|dZ(\lambda)|^p)^{q/p} = \text{const}(p, q, \alpha) \phi(\lambda) d\lambda$ for all $0 < p < \infty$, where the constant depends only on $p$ and $\alpha$ (and not on $Z$) and $\phi(\lambda)$ is a nonnegative integrable function called the spectral density of $X_t$. This spectral density plays a role, in linear inference problems, analogous to that played by the usual power density of $L^2$-stationary processes.

§8. Correspondence Principle.

Stochastic processes (Gaussian or $\alpha$-stable) are used in connection with problems such as estimation, detection, mutual information, economics, physics, etc. The problems are often effectively formulated in terms of corresponding measures on appropriate linear spaces of paths. Two important questions arising in this context are the following:

(a) Given a stable stochastic process with paths in a linear function space, is there a stable measure on the function space which is
induced by the given process?

(b) Given a stable measure on a linear function space, is there a stable process with paths in the function space which induces the given measure?

For a Gaussian processes the positive answers to both questions (a one-to-one correspondence) are well known cf. Rajput and Cambanis (1972), Rajput (1972), Byczkowski (1977, 1979), Vakhania (1979) and so on.

For stable processes only some answers to (a) were obtained in Cambanis, Miller (1980) for $L^p$ spaces $p > 1$ and in Rosinski, Tarieladze (1982) for so-called fundamental Banach spaces. Since we can find no reference* for a complete solution of the problem we prove in this section the correspondence theorem for stable processes for the function spaces $C[0,1]$ and $L^p[0,1]$.

Consider a stochastic process $X(t,w)$ as a family of r.v.'s. When the parameter set is ordered, then it is a mathematical model of some random phenomenon fluctuating in time. Other interesting sets for $T$ include $n$-dimensional Euclidean space $(n > 1)$ or an open connected subset of such space or even a Riemannian space. For a fixed $w$ we have a function $x(t,w)$ of $t$ which is called a sample path.

Viewing $\{X(t,w), t \in T\}$ as an infinite-dimensional random vector (r.v.) taking values in $\mathbb{R}^T$, its distribution is defined as follows

$$v_{t_1, \ldots, t_n}(C) = P(\langle X_{t_1}, \ldots, X_{t_n} \rangle \in B_n)$$

where $C$ is a cylinder subset of $\mathbb{R}^T$,

$$C = \{X \in \mathbb{R}^T; \langle x(t_1), \ldots, x(t_n) \rangle \in B_n\}.$$

Here $B_n$ is a Borel subset of $\mathbb{R}^n$.

If $\mathcal{F}^T$ denotes the smallest $\sigma$-field containing all cylinder sets, then Kolmogorov's consistency theorem says that a cylinder measure $v$ on $(\mathbb{R}^T, \mathcal{F}^T)$ is uniquely extendable to a probability measure $\mu$ iff it satisfies the consistency condition i.e., if the cylinder set $C$ has another expression

$$C_1 = \{x \in \mathbb{R}^T; \langle x(s_1), \ldots, x(s_m) \rangle \in B_m\}$$

*) After this work was completed, Prof. B.S. Rajput has informed us that his student D. Louis (Ph.D. Thesis; The University of Tennessee 1980) obtained a one-to-one correspondence between Banach space-valued measurable stable processes with a.a. sample paths belonging to $L_p(T,B)$ and stable measures on $L_p(T,B)$.
Thus the canonical model of a stochastic process can be described in the following manner: set $\Omega = \mathbb{R}^T$, $\mathcal{B} = \mathcal{C}^T$ and $P = \mu$. Then the relation $X(t,\omega) = x(t)$, $t \in T$, $x = x(t) \in \mathbb{R}^T$, clearly defines a stochastic process and its distribution is $\mu$.

The space $\mathbb{R}^T$, on which the distribution $\mu$ of the process is defined, is really quite a large space in general, and the subset which actually supports $\mu$ is often only a small part of it. For example, for Brownian motion $\mu$ is the Wiener measure on the space $C[0, +\infty)$ of all continuous functions on $[0, +\infty)$ (cf. Hida (1980), p. 46). For $\alpha$-stable motion (sometimes called also Levy stable motion or Levy process) the distribution $\mu$ is a $\alpha$-stable measure on the space $D[0, +\infty)$ of all right continuous functions which have finite left-hand limits (cf. Breiman (1968), p. 306).

The following proposition gives a correspondence between measurable processes and probability measures on the space $L^0(T,\Sigma,\mu)$ of all real measurable functions defined on $T$. $L^0(T,\Sigma,\mu)$ with the norm

$$
\|f\| = \int_T \frac{|f(t)|}{1 + |f(t)|} \, \mu(dt)
$$

which induces the topology of convergence in measure $\mu$, is a real Frechet space. If $\mu$ is nonatomic, then $L^0(T,\Sigma,\mu)$ has no nonzero continuous linear functionals. Recall that a stochastic process $X(t,\omega)$ is measurable if the map $(t,\omega) \rightarrow X(t,\omega)$ from $T \times \Omega$ into $\mathbb{R}$ is measurable relative to the product $\sigma$-field of $T \times \Omega$ and the Borel $\sigma$-field of $\mathbb{R}$.

**Proposition.** Let $A(T)$ be a separable Borel subspace of $L^0(T,\Sigma,\mu)$ endowed with the Borel $\sigma$-field induced by the topology of convergence in measure $\mu$. Every measurable stochastic process $X(t,\omega)$ with a.a. paths in $A(T)$ induces a probability measure on $A(T)$. Conversely, every probability measure $\mu$ on $A(T)$ is induced by a measurable stochastic process with a.a. paths in $A(T)$.

**Proof.** Define

$$
\tilde{X}(\omega) = \begin{cases} X(\cdot,\omega) & \text{if } X(\cdot,\omega) \in A(T) \\ 0 & \text{otherwise.} \end{cases}
$$

From the measurability of $X(t,\omega)$ and the separability of $A(T)$ endowed with the Borel $\sigma$-field induced by the topology of convergence in measure $\mu$ it follows that the mapping $\tilde{X}$ is a $A(T)$-valued r.v. The probability measure $\mu_\tilde{X}$ defined on $A(T)$ as the distribution of $\tilde{X}$ is just a probability measure induced by $X$.

Conversely, we will follow the construction in Byczkowski (1976). Since $A(T)$ is a separable metric space, then for each positive integer $n$ there
exists a countable collection \( S^{(n)}_k \) of Borel subsets in \( A(T) \) such that \( S^{(n)}_k \) are disjoint, \( \bigcup_{k=1}^{\infty} S^{(n)}_k = A(T) \) and \( \text{diam}(S^{(n)}_k) < \frac{1}{n} \). Moreover this collection can be chosen so that \( \{S^{(n+1)}_k\} \) refines \( \{S^{(n)}_k\} \). For each \( m \) choose an element \( h_m^{(n)} \in S^{(n)}_m \) and let \( h_m^{(n)} \) be a representative of the equivalence class \( h_m^{(n)} \). Define a sequence of stochastic processes \( X_n \):

\[
X_n(t,f) = \frac{h_m^{(n)}}{h_k^{(n)}}(t) \quad \text{if} \quad f \in S^{(n)}_k.
\]

It is easy to verify that each \( X_n : T \times A(T) \to \mathbb{R} \) is \((t,f)\)-measurable and that the sequence \( (X_n) \) is a Cauchy sequence i.e., for each \( \varepsilon > 0 \)

\[
m(t : |X_n(t,f) - X_k(t,f)| > \varepsilon) \to 0 \quad \text{if} \quad n,k \to \infty
\]

uniformly w.r.t. \( f \). Fubini's theorem implies that \( X_n \) is a Cauchy sequence in \( m \times \mu \) measure, and so, there exists a measurable stochastic process \( X : T \times A(T) \to \mathbb{R} \) such that \( X_n \to X \) in the \( m \times \mu \) measure. Fubini's theorem implies \( X_n \to X \) in \( m \) measure for \( \mu \)-almost all \( f \) and consequently

\[
X_n(t,f) + X(t,f) \mu-\text{a.s.}
\]

Let \( f \in A(T) \). Observe that for all \( n \), there exists \( k \) such that

\[
f \in S^{(n)}_k, \quad \text{and so,} \quad \|f - h_k^{(n)}\| < \frac{1}{n} \quad \text{and} \quad X_n(t,f) = \frac{h_k^{(n)}}{h_k^{(n)}}(t).
\]

Thus

\[
\hat{X}(f) = X(t,f) = f \mu-\text{a.s.}
\]

In particular, \( \mu_X = \mu \) and \( X(t,f) \in A(T) \) \mu-a.s.

**The correspondence theorem:** The following are true for \( A(T) = C[0,1] \) (or \( L^p[0,1] \), \( 0 < p < \infty \) or \( L^p\Phi[T] \)) and \( 0 < \alpha < 2 \).

(a) If \( (X_t)_{t \in T} \) is a strictly \( \alpha \)-stable (measurable) process with paths in \( A(T) \) then the map \( \hat{X} : (\Omega, \mathcal{F}) \to (A(T), B_A) \) defined by

\[
\hat{X}(\omega) = X(\cdot, \omega)
\]

is measurable and the probability \( \mu_X = P \cdot \hat{X}^{-1} \) induced on \( (A(T), B_A) \) is strictly \( \alpha \)-stable.

(b) If \( \mu \) is a strictly \( \alpha \)-stable measure on \( (A(T), B_A) \), there exists a strictly \( \alpha \)-stable (measurable) process \( (X_t)_{t \in [0,1]} \) with paths in \( A(T) \) which induces \( \mu \) on \( (A(T), B_A) \).

1° **Proof for the case** \( A(T) = C[0,1] \).

a) For the measurability of \( \hat{X} \) it suffices to show that \( f \circ \hat{X} \) is measurable for all \( f \in C[0,1]^* \). Also in order to show that \( \mu_X \) is strictly \( \alpha \)-stable it suffices to show that for all \( f \in C[0,1]^* \) \( f \circ \hat{X} \) is a strictly \( \alpha \)-stable r.v. on \( (C[0,1], B_{C[0,1]}, \mu_X) \), since strict \( \alpha \)-stability for all \( \alpha \) is a marginal property.
It should be noted that if a stochastic process has continuous paths for all \( w \in \Omega \), then it is product measurable.

Let \( f \in C([0,1]^\omega) \), then there exists a regular Borel measure \( \lambda \) on \([0,1]\) with compact support such that \( f(x) = \int_0^1 x(t) d\lambda(t) \) for all \( x \in C[0,1] \). It follows that there exists a real function \( g \) of bounded variation on \([0,1]\) such that \( f(x) = \int_0^1 x(t) dg(t) \) for all \( x \in C[0,1] \). Since \( x \)'s are continuous, we can write

\[
f(x) = \lim_{n \to \infty} \sum_{k=1}^n x(t_k) [g(t_k, n') - g(t_{k-1}, n')]
\]

for all \( x \in C[0,1] \), where \( t_{k,n} = a + (b-a)(k/n), k = 0,1, \ldots, n \). It follows from the definition of \( \hat{X} \) that

\[
(f \circ \hat{X})(w) = \lim_{n \to \infty} \sum_{k=1}^n X(t_k-n', w) [g(t_k, n') - g(t_{k-1}, n')]
\]

for all \( w \in \Omega \). Hence \( f \circ \hat{X} \) is measurable and also strictly \( \alpha \)-stable, since the a.s. limit of sequence of strictly \( \alpha \)-stable r.v.'s is a strictly \( \alpha \)-stable r.v.

b) Take \( (\Omega, B, P) \cong (C([0,1]), B_{C([0,1])}) \) and \( X(t,w) = w(t) \). The result is clear if we observe that the evaluation mapping \( \delta_t(x) = x(t) \) belongs to \( C([0,1])^\omega \) for all \( t \).

2° Proof for the case \( A(T) = L^\phi \)- the Orlicz space

By \( \phi \) let us denote a continuous, non-negative, non-decreasing function defined for \( u > 0 \) such that \( \phi(u) = 0 \) iff \( u = 0 \). Assume additionally that the function \( \phi(u) \) satisfies the so called \( \Delta_2 \)-condition i.e., there is a positive constant \( k \) such that \( \phi(2u) < k\phi(u) \). Let \( S \) be the space of equivalence classes of all real valued measurable functions with convergence in measure on an arbitrary \( \sigma \)-finite measure space \((T, E, m)\). Let \( L^\ast = L^\ast(m) \) be a separable subspace of \( S \). For \( x \in S \) let us put

\[
I_{\phi}(x) = \int_T \phi(|x(t)|) m(dt)
\]

and let \( L^\phi \) be the set of all \( x \in S \) such that \( I_{\phi}(ax) < \infty \) for a positive constant \( a \). The set \( L^\phi \) is a linear space under the usual addition and scalar multiplication. Moreover it becomes a complete linear space under the (usually non-homogeneous) seminorm \( I_{\phi} \cdot I_{\phi} = \inf\{c : c > 0, I_{\phi}(c^{-1}x) < c\} \). The space \((L^\phi, I_{\phi} \cdot I_{\phi})\) is called an Orlicz space. \( L^\phi \subseteq L^\ast \) is a separable Borel subset of \( L^\ast \) endowed with the Borel \( \sigma \)-field induced by the topology of convergence in measure \( m \). The best known examples of the Orlicz spaces are
(1) If $\phi(u) = \frac{u}{1+u}$, $T = [0,1]$ and $m$ is the Lebesgue measure on the $\sigma$-algebra of all Lebesgue measurable sets, then $L^\phi = L^\phi[0,1]$.

(2) If $\phi(u) = u^p$, $0 < p < \infty$, $T$ and $m$ as above, then $L^\phi = L^p[0,1]$. If $p > 1$ then $L^p$ is a Banach space, however for $p < 1$ $L^p[0,1]$ have no non-trivial continuous linear functionals.

Now we are able to prove the correspondence theorem for $A(T) = L^\phi(T)$.

a) Let $A > 0$ and $B > 0$ be given. Let $X^1 : \Omega \to L^\phi(T)$ and $X^2 : \Omega \to L^\phi(T)$ be independent r.v.'s with the same distribution as $\tilde{X}$. By the Proposition there exist measurable stochastic processes $X^1_t$ and $X^2_t$ with sample paths in $L^\phi(T)$ a.s. such that $\tilde{X}^1_t = X^1_t$ and $\tilde{X}^2_t = X^2_t$ a.s.. Since $X^1$ and $X^2$ have the same distribution, then, by Byczkowski (1977), there exists a $T_0 \in \Sigma$ with $m(T_0) = 0$ such that

$$<X^1_{t_1}, \ldots, X^1_{t_n}>, <X^2_{t_1}, \ldots, X^2_{t_n}> \text{ and } <X^1_{t_1}, \ldots, X^1_{t_n}>$$

have the same distribution and the first two r.v.'s are independent for all $t_1, t_2, \ldots, t_n \in T - T_0$. Thus since $X_t$ is strictly $\alpha$-stable, for each finite set $t_1, t_2, \ldots, t_n \in T - T_0$ we have that

$$A<X^1_{t_1}, \ldots, X^1_{t_n}> + B<X^2_{t_1}, \ldots, X^2_{t_n}> \text{ and } D<X^1_{t_1}, \ldots, X^1_{t_n}>$$

have the same distribution, where $D = (A^\alpha + B^\alpha)^\alpha$. It follows that $AX^1 + BX^2$ and $DX$ have the same distribution. Hence the measure $\nu_X = P \cdot X^{-1}$ induced on $L^\phi(T)$ is strictly $\alpha$-stable.

b) Conversely, let $\mu$ be a strictly $\alpha$-stable measure on $L^\phi(T)$. Then it follows easily from the classical definition that

$$(*) \quad \mu \times \mu((f,g) : (Af + Bg) \in S) = u(S)$$

for each $S \in B(L^\phi)$, $A > 0$, $B > 0$ and $C = (A^\alpha + B^\alpha)^\alpha$. Now let $X(t,f)$ be a measurable stochastic process with a.a. paths in $L^\phi$ constructed in the Proposition and corresponding to $\mu$. It follows from the construction that for each pair $(s,u)$ of reals

$$X(t,sf + ug) = sX(t,f) + uX(t,g) \text{ for } \mu_X \mu_X \text{ a.a.}(f,g,t).$$

Denote for simplicity $F(f,g) = C(Af + Bg)$ where $A, B$ and $C$ are as above. Thus the condition $(*)$ can be rewritten as $\mu_X(F^{-1}(S)) = u(S)$ for
Let $t_1, t_2, \ldots, t_n \in T$ be fixed and for $f \in L^\Phi$ consider the following r.v. in $\mathbb{F}^n$

$$\xi(f) = \langle X(t_1, f), \ldots, X(t_n, f) \rangle$$

If $u_\xi$ denote its distribution on $\mathbb{F}^n$ we want to show that $u_\xi$ is strictly $\alpha$-stable i.e., $u_{\xi} \mu_\xi(F^{-1}(\Delta)) = u_\xi(\Delta)$ where here $F$ is a corresponding mapping from $\mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}$ and $\Delta$ is any Borelian set on $\mathbb{F}$.

Namely, we have

$$\mu_\xi u_\xi(F^{-1}(\Delta)) = \mu \mu_{\xi}(\{(f, g) : \langle \xi(f), \xi(g) \rangle \in F^{-1}(\Delta)\}) =$$

$$\mu \mu_{\xi}(\{(f, g) : C(\mathbb{A}(f) + \mathbb{B}(g)) \in \Delta\}) =$$

$$\mu \mu_{\xi}(\{(f, g) : \xi(C(\mathbb{A}(f) + \mathbb{B}(g))) \in \Delta\}) =$$

$$\mu \mu_{\xi}(\{(f, g) : F(f, g) \in S_\Delta\}) =$$

$$\mu \mu_{\xi}(F^{-1}(S_\Delta)) = \mu(S_\Delta),$$

since $\mu$ is strictly $\alpha$-stable. Here $S_\Delta = \{f : \xi(f) \in \Delta\}$ which belongs to $B(L^\Phi)$ by the measurability of $\xi(f)$. Finally, $\mu(S_\Delta) = \mu(\{f : \xi(f) \in \Delta\}) = u_\xi(\Delta)$ and consequently $u_{\xi} \mu_\xi(F^{-1}(\Delta)) = u_\xi(\Delta)$. Hence $u_\xi$ is strictly $\alpha$-stable. Since the finite dimensional vector $\xi$ has been chosen arbitrarily in the span of the measurable stochastic process $X(t, f)$ it follows that the process is itself strictly $\alpha$-stable, which finishes the proof.

Remark: Let $T$ be any index set and $A(T)$ a topological vector space of real functions on $T$. Then a careful inspection of the above proof for $C[0,1]$ reveals that the correspondence principle also holds for $A(T)$ if the following sufficient condition is met: For every $t \in T$ the evaluation map $\delta_t(x) = x(t) \in A^*(T)$ and the linear span of $\{\delta_t, t \in T\}$ is weak* sequentially dense in the dual space $A^*(T)$. For example, one can take $A([a,b]) = C^n[a,b]$ - the separable Banach space of $n$-times continuously differentiable real functions on $[a,b]$ with the norm $1x1 = |x(a)| + \int^b_a|x'(t)|dt$.

Similarly, a careful inspection of the proof for $L^\Phi(T)$ shows that the correspondence principle holds for separable Borel subspaces $A(T)$ of $L^\Phi(T, L^\Sigma, m)$ endowed with the Borel $\sigma$-field induced by the topology of
§9. Stable Measures on Vector Spaces

Let $V$ be a vector space over $\mathbb{R}$ and $W$ be a $\sigma$-algebra of subsets of $V$. The pair $(V,W)$ is called a measurable vector space if

(i) addition is jointly measurable from $V \times V$ into $V$.

(ii) scalar multiplication is jointly measurable from $\mathbb{R} \times V$ into $V$.

A probability measure $\mu$ on a measurable vector space $(V,W)$ is called stable iff for any $A > 0$ and $B > 0$, and independent $V$-valued random variables $X$ and $Y$ with distribution $\mu$, there is $C > 0$ and $x \in V$ such that

$$\mathsf{LAW}(C(AX + BY) + x) = \mu.$$ 

The measure $\mu$ is strictly stable if we can always take $x = 0$ in the above formula. $\mu$ is symmetric iff $\mu(-S) = \mu(S)$ for any $S \in W$. It turns out that the constant $C$ always has the following form $C \equiv (\theta^a + 1^a)^{-1/\alpha}$ where $0 < \alpha < 2$, and the corresponding measure is called $\alpha$-stable.

It is known that for any Gaussian measure a linear subspace has measure 0 or 1. This zero-one law holds also for $\alpha$-stable measures.

**Theorem (Dudley, Kanter 1974)**

Let $\mu$ be a $\alpha$-stable measure on $(V,W)$. Let $E$ be a $\mu$ completion measurable linear subspace of $V$, then $\mu(E) = 0$ or 1.

The proof was simplified by Fernique (1974) and Smolenski (1981). As a corollary, one can show that for a separable $\mathcal{S}_\alpha\mathcal{S}$ process $X_t$ at every fixed $t \in T$, the paths of $(X_t)_{t \in T}$ are continuous or differentiable with probability zero or one. Also, if $(X_t)_{t \in T}$ is measurable, then with probability one its paths have essentially the same points of differentiability and continuity.

Using this observation the following characterization of absolute continuity of sample paths of $\mathcal{S}_\alpha\mathcal{S}$ process was obtained.

**Theorem (Cambanis, Miller 1980)**

Let $(X_t)_{t \in [a,b]}$ be a separable $\mathcal{S}_\alpha\mathcal{S}$ process with $1 < \alpha < 2$. Then each of the following two equivalent conditions is sufficient and necessary for the sample paths of $X_t$ to be absolutely continuous with probability one:

(i) The map $t \mapsto X_t$ is absolutely continuous
the covariation function $C_t = [X, Y]_t$ is absolutely continuous, its derivate $C'_t$ exists and $\int_T |X^0_t| \, dt < \infty$, where $Y$ is the linear span of the process, $t \in T - T_0$ and $X^0_t$ is the unique element from the span such that the covariation of $[X^0_t, Y]_t = C'_t$ for all $Y$.

**Corollary (Cambanis, Miller 1980)**

A separable harmonizable $\mathcal{S}$-process $X_t = \int e^{it \lambda} \, dZ_\lambda$ with the control measure $\mathbb{F}$ has absolutely continuous sample paths with probability one iff

$$\int \lambda^\alpha d\mathbb{F}(\lambda) < \infty, \text{ where } 1 < \alpha < 2.$$

**Theorem (Cambanis, Miller 1980)**

Let $(T, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and suppose that $L^p(T, \mathcal{F}, \mu)$ is separable, where $1 < p < \alpha$. Then for a measurable $\mathcal{S}$-process with $1 < \alpha < 2$ we have $\int |X_t|^p \, d\mu < \infty$ a.s. iff $\int |X_t|^p \, d|X_t|^p \, d\mu < \infty$.

The above result was obtained by an application of the following general theorem on tail behavior of stable measures. It extends the classical P. Levy's result on stable distributions on the real line, cf. §2.

**Theorem (de Acosta (1975))**

If $\mu$ is a $\sigma$-stable measure $0 < \alpha < 2$, and $q$ a measurable homogeneous seminorm on a measurable vector space $(V, W)$, then there exists a constant $C > 0$ such that

$$\mu\{x : q(x) > t\} < ct^{-\alpha}, \text{ for all } t > 0$$

In particular, for each $p < \alpha$

$$\int_V q^p(x) \, d\mu(dx) < \infty.$$

**Example**

Let $(T, \mathcal{F}, \mu)$ be a measurable space such that $\mu(T) < \infty$ and $L^p(T, \mathcal{F}, \mu)$ is separable. Let $X(t, \omega)$ be a measurable $\mathcal{S}$-process, $1 < \alpha < 2$. Define a new process by

$$Y(t, \omega) = X(t, \omega)(1 + E|X(t, \omega)|^p)^{-1/p} \text{ for } p < \alpha.$$

Note that
Thus the assumption \( m \) is finite and the Cambanis-Miller theorem imply that
\[ Y(t,\omega) \in L^p(T,\Sigma,m) \quad \text{a.s.} \]

Define the mapping \( M : L^p(T,\Sigma,m) \rightarrow L^p(T,\Sigma,m) \) by the formula
\[ (Mf)(t) = f(t)(1 + C_X(t))^{1/p}, \]
where \( C_X(t) = \mathbb{E}|X(t,\omega)|^p \). (Of course, \( C_X(t) \)
can be also described by the covariation function of the process, since
\[ (\mathbb{E}|X(t,\omega)|^p)^{1/p} = c(p,\alpha)[X(t,\omega),X(t,\omega)]_{\alpha}^\alpha. \]
It is easy to see that \( M \) is linear and continuous mapping and \( MY(t,\omega) = X(t,\omega) \).

By the de Acosta theorem a SnS measure \( \mu_Y \) induced by a measurable SnS process \( Y \) with a.s. paths in \( L' \) satisfies the inequality
\[ \mu_Y(f : \|f\|_{L'}, > t) < c_1 t^{-\alpha}. \]

As above for any measurable SnS process \( X \) there exists a measurable SnS process \( Y \) with a.a. paths in \( L' \) such that
\[ Y(s,\omega) = \frac{X(s,\omega)}{1 + \mathbb{E}|X(s,\omega)|}. \]
Thus there exists \( c > 0 \) such that
\[ \mu_X(f : q_0(f) > t) < ct^{-\alpha} \]
for the probability measure \( \mu_X \) on \( L' \) induced by the SnS process \( X \), where \( q_0 \) denotes the non-homogeneous seminorm
\[ q_0(f) = \int_T \frac{|f|}{1 + |f|} \, dm \quad \text{in} \quad L^p(T,\Sigma,m). \]

For \( \alpha = 2 \) this result is known as the Ryll-Nardzewski example, see Byczkowski (1976). In this case, since all moments of a Gaussian process are finite, one can choose \( p = 2 \). Of course, for Gaussian processes the last estimation should be changed since the tail behaviour is very different.

From the de Acosta (1975) theorem one has
\[ \limsup_{t \rightarrow \infty} \mu_X(x : q(x) > t) < \infty. \]
This result was improved next to
Theorem (de Acosta (1977))

Under the conditions of the previous theorem the limit
\[ \lim_{t \to \infty} t^\alpha \mu \{ x : q(x) > t \} \]
exists and is strictly positive if \( \alpha < 2 \) and \( \mu \) satisfies the nondegeneracy condition.

Thus a complete generalization of P. Levy's result on the real line holds in the most general setup of measurable vector spaces.

If \( E \) is a Hausdorff topological space, \( \mathcal{B} \) the Borel \( \sigma \)-algebra of \( E \) then the support of a probability measure on \( (E, \mathcal{B}) \) is the set
\[ S(\mu) = \{ x \in E : \mu(U) > 0 \text{ for every open set } U \text{ containing } x \} \]

It is well known that \( S(\mu) \) is closed and if \( \mu \) is Gaussian, then \( S(\mu) \) is a closed subspace of \( E \). De Acosta (1975) has proved that if \( \mu \) is \( \mathcal{S}_\alpha \) measure with \( \alpha > 1 \) then this same holds. Independently, a similar result for \( \tau \)-regular \( \mathcal{S}_\alpha \) measures \( (\alpha > 1) \) on locally convex topological vector spaces was obtained by Rajput (1977). In the next paper he has solved problem completely.

Theorem (Rajput, (1977 a))

Let \( \mu \) be a \( \tau \)-regular \( \mathcal{S}_\alpha \), \( 0 < \alpha < 2 \), (or even infinitely divisible) measure on a locally convex topological vector space \( E \). Then \( S(\mu) \) is a closed subspace (respectively subgroup) of \( E \).

Let \( v \) be a \( \mathcal{S}_\alpha \) cylinder measure and \( \mu \) a Radon \( \mathcal{S}_\alpha \) measure on a Banach space \( E \). If for all \( a \in E^* \) the inequality
\[ |1 - \hat{v}(a)| < |1 - \hat{\mu}(a)| \]
holds, then is \( v \) Radon too? This result is true for \( \alpha = 2 \) and is false for all \( \alpha < 2 \) and a class \( V \) of spaces for which (and only for which) the above implication holds was introduced in Tien and Weron (1980). See also, Mandrekar and Weron (1982).

Linde and Mathe (1980) have invented a class of Banach spaces \( A_\alpha \) for which the following holds: there exists a \( c > 1 \) such that for all \( a \in E^* \)
\[ \int_E |\langle x, a \rangle|^r d\nu(x) < \int_E |\langle x, a \rangle|^r d\mu(x) \]
implies
for all $\sigma$-finite measures on $E$. Does $A_a = V_a$? This problem was first raised by the author during the 3rd Vilnius Conference 1981. An interesting result in this direction has been obtained by Mathe (1982), see also Linde (1983), but the problem is still open. This question is especially interesting for $\sigma$-finite measures generated by $\sigma$-finite processes with a.a. sample paths belonging to some concrete function space. This suggests the following problem: Find a relationship between $A_a, V_a$ classes and an analogue of Slepian's lemma for $\sigma$-finite processes. See §6 for more details.

§10. Levy's Representation

A representation of the Fourier transform for $\sigma$-finite measures on any real separable Hilbert space was obtained by Kuelbs (1973). This result is an improvement of that obtained first by Jajte (1968) and Kumar and Mandrekar (1972).

Theorem (Levy's Spectral Representation)

(i) If a probability measure $\mu$ on a separable Banach space $E$ is $\alpha$-stable, $0 < \alpha < 2$, then its characteristic functional can be written in the form:

\[
\hat{\mu}(a) = \exp\left(-\int \frac{|<x,a>|^{\alpha}d\Gamma(x) - \text{ic}(\alpha,a) + i<x,a>}{U}\right)
\]

where $\Gamma(\cdot)$ is a finite measure on the unit sphere $U = \{x : \|x\| = 1\}$ of the Banach space $E$, $x$ is a vector in $E$ and the function $c(\alpha,a)$ is given by

\[
c(\alpha,a) = \begin{cases}
\tan \frac{\pi \alpha}{2} \int_{U} |<x,a>|^\alpha \text{sign}<x,a>d\Gamma(x) & \text{if } \alpha \neq 1 \\
2/\pi \int_{U} \log|<x,a>|d\Gamma(x) & \text{if } \alpha = 1
\end{cases}
\]

(ii) Conversely, in Banach space of type $p$, $1 < p < 2$, any functional of the form $(\cdot)$ with $\alpha < p$ is the characteristic functional of an $\alpha$-stable measure on $E$.

This theorem subsumes the results of many authors, Tortrat (1976), Dettweiler (1976), (where a case of more general vector spaces was considered), Paulauskas (1976, 1978), Jurek and Urbanik (1978), Marcus and Woyczynski (1978) etc. For the proof we refer the readers to Linde (1983).
Since a Hilbert space has type 2, the above theorem also contains Kuelb's result. Call the measure $\Gamma$ in (*) the spectral measure on the $\alpha$-stable measure $\mu$. Only in Banach spaces of $\alpha$-stable type (i.e., type $\alpha + \epsilon$ for some $\epsilon > 0$) any finite measure on $U$ is spectral. However a description of all spectral measures of stable measures is known for $L^p$, Orlicz spaces (Rackauskas (1980)) and $C[0,1]$. The last space is important, since $\alpha$-stable measures on $C[0,1]$ correspond to $\alpha$-stable processes on $[0,1]$ with continuous paths. Let's mention only that such processes do not have independent increments, as the last have paths in $D[0,1]$.

The following sufficient conditions for any finite measure $\Gamma$ on $C[0,1]$ to be a spectral measure of a $\alpha$S measure $\alpha > 1$ were obtained by Araujo (1975) and Paulauskas (1978).

Theorem: If one of the following holds then $\Gamma$ is a spectral measure of a $\alpha$S measure on $C[0,1]$.

1. There exists constants $C_\Gamma$ and $\gamma > 1$ such that

$$\int_{U} |x(t) - x(s)|^\gamma \Gamma(dx) < C_\Gamma |t - s|^\gamma$$

2. \[
\int_0^1 \int_0^1 \exp\left(\int_0^t \left[ \exp<r(z) - z(t)\rangle - 1 - \frac{r(z) - z(t)}{\rho(u - t)} \right] \frac{dr}{1 + \rho(z)} dz \right) du dt < \infty
\]

where $\rho$ is continuous increasing function on $[0,1]$, $\rho(0) = 0$, $\rho(u) = \rho(-u)$ for $-1 < u < 0$ and $\int_0^1 \rho(u) u^{-1} du < \infty$.

3. The support of $\Gamma$ is contained in the class Lip $\delta$, $\delta > 0$ of the function, satisfying the Lipschitz condition with exponent $\delta$.

Example

Observe that if the condition (i) of the previous theorem holds then the corresponding $\alpha$S process $X_t$ has continuous sample paths and moreover

$$\mathbb{E} \sup_{0 \leq s, t \leq 1} |X(t) - X(s)| < C(\Gamma, \alpha, \delta, \gamma)h^{\frac{\gamma\delta - 1}{\alpha\delta}},$$

where $\delta$ is such that $\alpha \geq \delta > \frac{\alpha}{\gamma}$. If for example $\gamma = \alpha$, then sample paths of $X_t$ are absolutely continuous and

$$\mathbb{E}|X'(t)|^{\delta} < C(\Gamma, \alpha, \delta)$$

for $\delta < \alpha$. 
In the Gaussian case a symmetric measure \( \mu \) has a series expansion in any separable Banach space \( E \) i.e., there exists an \( E \)-valued random vector \( x \) with the Gaussian distribution \( \mu \) such that \( x = \sum_{n=1}^{\infty} \gamma_n(z_n) \); \( \gamma_n \) - are i.i.d. standard Gaussian \( a_n \in E \) and the series is a.s. convergent in the norm of \( E \), see Jain and Kallianpur (1970) or LePage (1972). In sharp contrast with the Gaussian case this is no longer true for \( a < 2 \).

EXAMPLE

Consider on \( E = \mathbb{R}^2 \) a measure defined by its characteristic functional

\[
\hat{\mu}(a) = \exp\left(-\left|a_1\right|^2 + \left|a_2\right|^2/2\right), \quad a = (a_1,a_2).
\]

Then it is a S\(\alpha\)S measure for \( 1 < a < 2 \).

If \( \mu \) has a series expansion, then there exists a sequence \( \{x_i\} \subseteq \mathbb{R}^2 \) such that \( x = \sum_{i=1}^{\infty} x_i \theta_i \) is a.s. convergent in \( \mathbb{R}^2 \) where \( \theta_i \) - are i.i.d. standard S\(\alpha\)S r.v.'s and the distribution of \( x \) is \( \mu \). This implies for all \( a \in \mathbb{R}^2 \)

\[
\hat{a}! = (\sum_{i=1}^{\infty} \langle x_i, a \rangle)^{1/a}.
\]

Thus the mapping \( a \mapsto \{\langle x_i, a \rangle\} \) is an isometric embedding of \( \ell_2^\alpha \) into \( \ell_\alpha \), but it is known that it cannot be true, see for example Linde (1983) p. 113.

EXAMPLE

It is easy to observe that such series expansion holds for a S\(\alpha\)S - measure with a discrete spectral measure \( \Gamma \) (i.e., \( \Gamma = \sum_{i=1}^{\infty} a_i \delta_{x_i} \), \( a_i < \infty \), and \( \delta_{x_i} \) a Dirac measure concentrated at point \( x_i \in E \)). In such case the spectral measure has the form

\[
\frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j} x_j^{\alpha} \delta_{x_j} + \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j} x^{-\alpha}_{-j} \delta_{x_j}.
\]

Compare also with the problem of series representation for S\(\alpha\)S processes discussed in §4.

§11. Weak Convergence

The most frequently used tool for the purpose of characterization of stable measures on real, separable Hilbert and Banach spaces is an appropriate version
of the convergence of types theorem, cf. Jajte (1968), Kumar and Mandrekar (1972). A general version of such result was obtained by Csiszar and Rajput (1976) for probability measures on arbitrary topological vector spaces.

**Theorem (Csiszar, Rajput (1976)).**

Let \( \{ \mu_n \} \) be a sequence of \( \tau \)-regular stable measures on a topological vector space, weakly converging to some non-degenerate \( k \)-regular measure \( \mu \). Then \( \mu \) is stable. Moreover, if the \( \mu_n \)'s are strictly stable then so is \( \mu \), even under the weaker hypothesis \( \mu - \) is \( \tau \)-regular.

Let us recall that \( \mu \) is \( k \)-regular if \( \mu(A) = \sup_{k \in A} \mu(k) \) for every Borel set \( A \), where \( k \) ranges over the compact subsets of \( A \); \( \mu \) is \( \tau \)-regular if \( \mu(G) = \lim \mu(G_u) \) for every increasing net of open sets \( G_u, G \).

Let \( \{ \mu_u \} \) be a family of SøS measures on a separable Banach space \( E \) with characteristic functionals

\[
\mu_u^\wedge(x^*) = \exp\left(- \frac{1}{\alpha} \left| \langle x, x^* \rangle \right|^\alpha \Gamma_u(x) \right), \quad x^* \in E^*, \quad 0 < \alpha < 2,
\]

where \( S \) is the unit sphere of \( E \) and \( \Gamma_u \) is a corresponding spectral measure, see §10.

**Theorem (N.Z. Tien (1982))**

The following conditions are equivalent:

1) \( E \) is of \( \alpha \)-stable type

2) \( \{ \mu_u \} \) is uniformly tight iff so is \( \{ \Gamma_u \} \)

3) \( \{ \mu_u \} \) converges weakly to \( \mu \) iff so does \( \{ \Gamma_u \} \)

**Theorem (Linde 1983)**

Let \( \{ \mu_n \} \) be a sequence of \( \alpha_n \)-stable, \( 0 < \alpha_n < 2 \), measures with \( \mu_n \) weakly converging to \( \mu \). If \( \sup_n \alpha_n < 2 \), then the corresponding sequence of spectral measures \( \{ \Gamma_n \} \) converges weakly to the spectral measure of \( \mu \).

The preceding fact becomes false without the assumption \( \sup_n \alpha_n < 2 \), since if \( \alpha_n \rightarrow 2 \) then \( \mu \) becomes a Gaussian measure.

**Theorem (Rosinski 1982)**

Let \( \mu \) be a SøS measure on a separable Banach space. Then there exists a sequence of SøS measures \( \mu_n \) with the finite support spectral measures \( \Gamma_n \) such that \( \{ \mu_n \} \) converges weakly to \( \mu \).
§12. Random Integrals w.r.t. Stable Measures

As one of the useful tools in a study of stable measures Marcus, Woyczynski (1979) and Okazaki (1979) have employed the random integrals of Banach space valued functions w.r.t. SgS random measure. If \((T,\mathcal{E},\nu)\) is a finite measure space then by a SgS random measure \(M\) we mean an independently scattered \(\sigma\)-additive set function \(M : \mathcal{E} \to L(\Omega)\) such that

\[
\mathbb{E} \exp i \ u \ M(A) = \exp(-\nu(A)|u|^\alpha),\quad u \in \mathbb{R}.
\]

The random integral w.r.t. \(M\) was defined as a continuous operator on \(L_{\mathbb{E}}^\alpha\) and exists only when the Banach space \(\mathbb{E}\) has \(\sigma\)-stable type. Such integrals without any restrictions on geometry of \(\mathbb{E}\) are studied by Rosinski (1982). He extended the earlier results of Urbanik and Woyczynski (1967) on random integrals for real valued functions.

**Theorem (Rosinski 1982)**

Let \(\mathbb{E}\) be a Banach space and \(M\) be standard SgS Random measure on \((T,\mathcal{E},\nu)\). Then an \(\mathbb{E}\)-valued function \(f\) is \(M\)-integrable i.e., \(f \in L_{\mathbb{E}}^\alpha(M)\) iff 

\[
f\text{ is strongly measurable and the function } \phi_f(x^*) = \exp(- \int_{T} |<x^*, f>|^\alpha \nu)
\]

is the Fourier transform of a Radon probability measure on \(\mathbb{E}\). In this case the Fourier transform of the law of \(\int fdM\) is equal to \(\phi_f\). Moreover 

\[
L_{\mathbb{E}}^\alpha(M) \subset L_{\mathbb{E}}^\alpha(\nu)\quad \text{and equality holds for } \quad 0 < \alpha < 1.
\]

Using the above result one can deduce integral representation for any SgS measure \(\mu\) on \(\mathbb{E}\). Namely, there exists an \(\mathbb{E}\)-valued function \(f \in L_{\mathbb{E}}^\alpha(M)\) on an atomless measure space \((T,\mathcal{E},\nu)\) such that

\[
\text{LAW } \int_{T} f \, dM = \mu.
\]

This is an analogue of the celebrated spectral representation established for SgS processes by Bretagnolle et al (1965). See §3 for more details.

The following stronger result is an analogue of the minimal spectral representation for a SgS process from §3 and can be proved exactly in the same way. (cf. C.D. Hardin, Jr. (1982), pp. 388-390).

**Theorem**

Let \(\mu\) be a SgS measure on a separable Banach space \(\mathbb{E}\) and let \(M\) be a SgS random measure on \(([0,1],\mathcal{B})\) with \(\mathbb{E} \exp(it \ M(0,s)) = \exp(-s|t|^\alpha)\). There exists a minimal representation for \(\mu\) i.e., there exists a set of functions
$\mathcal{F} = (f_\star, x \in E^*) \subset L^\alpha([0,1])$ such that:

(i) there is no set $B \subset [0,1]$ of positive measure such that $f_\star = 0$ a.e. on $B$ for all $x \in E^*$

(ii) corresponding to every Borel set $B$ which is almost disjoint from the atoms of $\rho(\mathcal{F}) = \sigma(f/g)$, $f,g \in \mathcal{F}$, there is a set $B' \in \rho(\mathcal{F})$ such that $|B \Delta B'| = 0$

(iii) whenever $B$ is an atom of $\rho(\mathcal{F})$, then $f_\star$ is a.e. constant on $B$ for all $x$

(iv) an $E$-valued random element $x$ defined for $x \in E^*$ by

\[ x^*(x) = \int f^*(s)M(ds) \]

Moreover, if $x^* + f_\star$ and $x^* + g_\star$ are minimal representations for a given non-Gaussian SaS measure (i.e., they satisfy (i)-(iv)), then there exists an isometric automorphism $V$ of $L^2[0,1]$ such that $Vf_\star = g_\star$ for all $x \in E^*$.

Now we would like to present some interesting properties of such random integrals for the case $E = \mathbb{R}$.

**Theorem**

The operator $\mathbb{M}: f \rightarrow \int f \, dM$ defined by the random integral w.r.t. a SaS random measure $M$ has the following properties:

(i) $\mathbb{M}(f)^r_t = c(r,\alpha)\|f\|_\alpha \quad \forall f \in L^\alpha$ and $0 < r < \alpha$,

(ii) if $\mathbb{M}(f_n)$ converges to $\mathbb{M}(f)$ in $L^\alpha$, then $\lim_{n} f_n = f$ in $L^\alpha$,

(iii) the r.v.'s $\mathbb{M}(f_1)$ and $\mathbb{M}(f_2)$ are independent iff $\nu\{f_1f_2 \neq 0\} = 0$ for $0 < \alpha < 2$,

(iv) if $1 < \alpha < 2$ then $\mathbb{M}(f_2)$ is James orthogonal to $\mathbb{M}(f_1)$ iff $[f_1, f_2]_\alpha = 0$ cf. the end of §1,

(v) $E(\mathbb{M}(f_1)\mathbb{M}(f_2)) = [f_1, f_2]_\alpha \cdot \mathbb{M}(f_2)$ for $1 < \alpha < 2$,

(vi) If $1 < \alpha < 2$ and $0 < r < \alpha$ then the following diagram commutes:

\[
\begin{array}{ccc}
L^\alpha(\Omega_1, \Sigma_1, \mathbb{P}_1) & \xrightarrow{\mathbb{M}} & L^r(\Omega_2, \Sigma_2, \mathbb{P}_2) \\
E(\cdot | \Sigma_1^0) & + & E(\cdot | \Sigma_2^0) \\
L^\alpha(\Omega_1, \Sigma_1, \mathbb{P}_1) & \xrightarrow{\mathbb{M}} & L^r(\Omega_2, \Sigma_2, \mathbb{P}_2)
\end{array}
\]
where $\Sigma_0^1$ is a sub $\sigma$-field of $\Sigma_1$ and $\Sigma_2^0 = \{ M(A) : A \in \Sigma_1^0 \}$.

The operator $M$ defines an embedding of $L^0(\mathcal{F}_1, \Sigma_1)$ into $L^r(\mathcal{F}_2, \Sigma_2)$ and is called the stable embedding of $L^\alpha$ into $L^r$ for $0 < r < \alpha$. The above theorem subsumes the results of many authors. For the proof and additional informations see Linde (1983).

In the fundamental Wiener-Ito decomposition of $L^2$, $L^2 = \sum_{n=0}^{\infty} H^n$, the space $H^n$ is referred to as the multiple Wiener integral of degree $n$ and the elements of $H^n$ may be regarded as members of $L^2$ which can be expressed as $n$-fold integrals w.r.t. Brownian motion, see Hida (1980). From a glance at this book is clear that a basic question in order to develop the causal calculus (Hida type theory) for $\alpha$-stable noise, is to define an analogue of the multiple Wiener integral for $\alpha$-stable motion. As yet only some rather preliminary results, going in this direction, are known. cf. also Taqqu and Wolpert (1983).

Let $M$ be an $\alpha$-stable random measure on Borel subsets of $[0,1]$ such that its Fourier transform has the form $M(A) = \exp(-|A|\alpha|t|^{\alpha})$, $0 < \alpha < 2$, where $|A|$ denotes the Lebesgue measure of $A$.

Theorem (Surgailis (1983))

Let $r < \alpha < q$ and assume that

$$\sum_D \left( \int_{[0,1]^n} |f(t_1, \ldots, t_n)|^q (dt) |D|^r/q (|dt|)^{n-|D|} \right)^{1/r} < \infty$$

where the sum extends over all subsets $D \subset \{1, 2, \ldots, n\}$. Then the integral $\int f \, dM^n$ exists (as an unconditional limit in $L^r$ of approximate sums).

Theorem (Szulga, Woyczynski (1983))

If a kernel $f(s,t)$ has the orthogonal expansion

$$f(s,t) = \sum_{j,k} c(j,k) \phi_j(s) \phi_k(t),$$

where $(\phi_j)$ is the $L^\alpha$-normalized Haar system and if

$$\sum_{j,k} |c(j,k)|^{\alpha/2} < \infty$$

then the integral

$$\int f(s,t) M(ds) M(dt) = \sum_{j,k} c(j,k) X_j X_k$$

exists, where $X_j = \int \phi_j(s) M(ds)$.
Theorem (Rosinski, Woyczynski (1983))

If a kernel \( f(s,t) \) satisfies the condition
\[
\frac{1}{q} \int \left( \frac{1}{q} \int |f(s,t)|^q ds \right)^{\alpha/q} dt < \infty
\]
for \( \alpha > q \), then
\[
\int \left( \int f(s,t) M(dt) \right) M(ds)
\]
exists.

§13. Radonifying Operators

If \( E \) is a Banach space, then a linear operator \( F : L^\beta \to E \) is called \( \alpha \)-radonifying if \( \exp(-\|F^* y\|_q^q), y \in E \) and \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \), is the characteristic functional of a Radon measure on \( E \). Denote by \( \Sigma_\alpha(L^\beta, E) \) the class of all such operators, \( 0 < \alpha < 2 \). The class \( \Sigma_2(L^2, E) \) - Gauss-radonifying operators - has been extensively studied, see, Linde and Pietsch (1974). The main result is that \( \Sigma_2(L^2, E) \) coincides with the class \( \pi_2(L^2; E) \) of all 2-absolutely summing operators iff \( E \) is of cotype 2, cf. Chobanian and Tarieladze (1977). For \( 1 < \alpha < 2 \) \( \theta \)-radonifying operators were studied in Linde, Mandrekar and Weron (1980), where the following preliminary fact was proved:

Proposition

Let \( 1 < \alpha < 2 \). The following are equivalent:

a) \( F \in \Sigma_\alpha(L^\beta, E) \)

b) \( \exp(-\|F^* y\|_q^q) \) is the characteristic function of a Radon measure on \( E \).

c) \( F^* \) is decomposable i.e., there exists an \( E \)-valued strongly measurable r.v. \( X = F \mathbb{0} \) such that \( F^* y = y^*(x) \).

Moreover, if \( L^\beta \) is replaced by \( L^\beta \), then a,b, and c are equivalent to

d) the series \( \sum \mathbb{E} e_i \mathbb{0} \) converges a.s. in \( E \), where \( (e_j) \) is the standard basis in \( L^q \).

It turns out that, in general, neither \( \Sigma_\alpha(L^\beta, E) \subset \pi_\alpha(L^\beta, E) \) nor the converse inclusion hold. It has been proved that \( \pi_\alpha(L^\beta, E) \subset \Sigma_\alpha(L^\beta, E) \) iff \( E \) is of \( \alpha \)-stable type and that \( \pi_\alpha(L^\beta, E) \supset \Sigma_\alpha(L^\beta, E) \) iff \( E \) is of \( \alpha \)-stable type and \( E \) is isomorphic to a subspace of a quotient of some \( L^\alpha \)-space, see Linde, Mandrekar, Weron (1980), th. 3.
From the point of view of operator ideals it is convenient to study the dual ideal $\Lambda_\alpha(L^\alpha, L^\beta) \equiv \Sigma^\text{dual}(L^\beta, L^\alpha)$ see Linde (1982 and 1983).

**Theorem** (Linde, Mändrekar, Weron (1980); Thang, Tien (1980)).

Suppose $1 < \alpha < 2$. Then the following are equivalent

(i) $E$ has $\alpha$-stable type and is isomorphic to a subspace of some $L^\alpha$

(ii) $\Lambda_\alpha(L^\alpha, L^\beta) = \pi_\alpha(L^\alpha, L^\beta)$

A complete description of the ideal $\Sigma(L^\beta, E)$ or equivalently $\Lambda_\alpha(L^\alpha, L^\beta)$ is only known in some concrete Banach spaces. If $E = E^p$ see Rackauskas (1979) and for $E = L^p$ see Linde, Mändrekar, Weron (1980). In both papers $\alpha \neq p$. The case $\alpha = p$ was described by Kwapien (1981), see also Cambanis, Rosinski and Woyczynski (1983), and Gine, Zinn (1983).

**Theorem** (Kwapien 1981)

If $1 < \alpha < 2$ then $F = (f_{jk}) \in \Lambda_\alpha(L^\alpha, L^\beta)$ iff the following series is finite

$$\sum_{j,k} |f_{jk}|^\alpha (1 + |\log \frac{|f_{jk}|^\alpha}{\sum_n |f_{jn}|^\alpha |f_{nk}|^\alpha}|)$$

For a direct proof of this result, as well as, for some results on the convergence of the random quadratic forms $E f_{jk} \Theta_j \Theta_k$, where $(f_{jk})$ is a real infinite matrix and $(\Theta_k)$ is a sequence of i.i.d. $\mathcal{N} \sigma$ r.v.'s the reader is referred to. Cambanis, Rosinski and Woyczynski (1983). Let's mention only that the convergence of such random quadratic forms have implications in double Wiener-type random integrals discussed in §12. Namely, we have the following.

**EXAMPLE**

Let $f(s,t) = \sum_{j,k} a_{jk} A_j(s,t)$ where $A_j = [a_{j,j+1}]$

$0 = a_1 < a_2 < \ldots < a_j < 1$, $j = 1, 2, \ldots$. Then the iterated integral

$$\int_0^t f(s,t)M(ds)M(dt),$$

where $M$ is a $\alpha$-stable random measure, exists iff

$$\int_0^t |f(s,t)|^\alpha (1 + |\log \frac{|f(s,t)|^\alpha}{\int_0^t |f(s,u)|^\alpha du \int_0^t |f(u,t)|^\alpha du}|)dsdt < \infty$$
Observe that putting $g_{jk} = b_j |A_j|^{1/\alpha} |A_k|^{1/\alpha}$

$$\int_0^t \int_0^s f(s, t) M(ds) M(dt) = \sum_{j<k} g_{jk} G_j G_k$$

one can see a relation with Kwapien's condition. For more details, see Cambanis, Rosinski and Woyczynski (1983).

§14. Operator Stable and Semistable Measures

In the last section we consider two natural generalizations of stable distributions.

Let $X_1, X_2, \ldots$ be i.i.d random vectors with partial sums $S_n = X_1 + \ldots + X_n$. It is natural to ask how the distributions of the partial sums $S_n$ can be asymptotically approximated. Classically the approximation has been made by finding sequences $a_n$ and $b_n$ of norming constants and centering vectors such that $\text{LAW}(S_n/a_n - b_n)$ converges weakly to a limit law $\nu$. Specializing to i.i.d. summands, the central limit problem reduces to the following three:

1. characterization of the possible limit laws,
2. characterization of the generalized domain of attraction of a limit law $\nu$; i.e., characterization of those laws whose partial sums can be affinely normed to converge to $\nu$, and
3. construction of canonical affine norming transformations.

Levy (1937) found a complete solution to these three problems in $\mathbb{R}$, as represented by the stable laws. Sharpe (1969) settled problem (1) in Euclidean space $\mathbb{R}^d$ introducing the operator stable laws.

In order to define them let us recall that vector $X$ in $\mathbb{R}^d$ is called full if $\mathbb{R}^d$ is the linear span of the support of $\text{LAW}(X)$. Given a full limit law $\mu$ on $\mathbb{R}^d$, we are interested in identifying the collection of random vectors $X$ on $\mathbb{R}^d$ for which there exist centering vectors $v_n \in \mathbb{R}^d$ and norming linear operators $A_n$ on $\mathbb{R}^d$ such that the law of $A_n(S_n - v_n)$ converges weakly to $\mu$ as $n \to \infty$.

This collection of random vectors $X$ forms the generalized domain of attraction (GDOA) of $\mu$. The limit laws $\mu$ which have a nonempty GDOA are called operator-stable laws. The operator-stable laws are a subclass of the infinitely divisible laws which properly contains the stable laws.

Under Urbanik's (1973) condition on the norming transformations Krakowiak (1979) extended Sharpe's results to Banach spaces. the DOA of any operator-stable law in $\mathbb{R}^d$ was characterized by Jurek (1980) and Hudson-Mason-Veh
Hahn and Klass (1980, 1981, 1983) solved problems (2) and (3) for spherically symmetric limits on \( \mathbb{R}^d \) and have characterized recently the GDOA of every operator-stable law on \( \mathbb{R}^d \). For a comprehensive survey of the work on the domain of attraction of stable distributions in Banach spaces, cf. also Mandrekar (1981).

Another natural and nontrivial generalization of stable measures is the class of r-semistable measures, which was first introduced and studied on real line \( \mathbb{R} \) by Levy (1937). Later Kruglov (1972) obtained a quite explicit form of the characteristic function of r-semistable measures on Hilbert spaces. The general zero-one dichotomy theorem for r-semistable laws on infinite dimensional linear spaces was established in Louie-Rajput-Tortrat (1980). It includes Dudley-Kanter theorem for stable measures.

Let \( 0 < r < 1 \) and let \( E \) be a real separable Banach space. A Borel probability measure \( \mu \) on \( E \) is called r-semistable if there exist \( x_n \in E, a_n > 0, \) a sequence \( k_n \) of positive integers and a Borel probability measure \( v, k_n^{-1} x_{n+1} + r \) and \( v_n^{*} m^{-1} \delta_{x_n} \) converges weakly to \( \mu \) as \( n \to \infty \). It is shown in Chung-Rajput-Tortrat (1982) that \( \mu \) is r-semistable iff \( \mu \) is infinitely divisible and

\[
\mu^{*n} = \mu * m^{-1} \delta_{x_n}, \quad u = r^{n/\alpha}, \quad 0 < \alpha < 2.
\]

Recently, Rajput and Rama-Murthy (1983) have introduced the notion of semistable processes, defined the stochastic integrals w.r.t. semistable random measures, and obtained the spectral representation of not necessarily symmetric semistable and stable processes. In this way they enlarge the scope of the spectral representations, see §3, to a larger class of processes including those studied in Bretagnolle et al (1966), Kuelbs (1973) and Schilder (1970). Crucial for this result is a characterization of semistable probability measures on a separable Banach space \( E \) in terms of a finite measure on a suitable ring of \( E \). This measure on the ring plays here similar role as did the spectral measure \( \Gamma \), defined on the unit sphere of \( E \), of the symmetric stable measure.

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SAMPLE PATHS OF DEMIMARTINGALES
Thomas E. Wood

Let $(\Omega, F, P)$ be a probability space and $S_1, S_2, \ldots$ a sequence of random variables in $L^1(\Omega, F, P)$. Newman and Wright (1982) defined this sequence to be a demi(sub)martingale if whenever $f$ is a (nonnegative) function of $n$ variables, nondecreasing in each variable separately, it follows that

$$E(f(S_1, \ldots, S_n)(S_{n+1} - S_n)) \geq 0.$$ 

Both martingales and partial sum processes of the form $S_n = \sum_{i=1}^{n} X_i$ where $X_1, X_2, \ldots$ are associated, mean zero, random variables are demimartingales.

In this paper we extend classical martingale results, such as the martingale convergence theorem, to demimartingales, as well as discuss some examples. The upcrossing lemma and maximal inequality, in somewhat different forms, appear previously in the paper of Newman and Wright. In their work, the maximal inequality provides tightness of distributions of partial sum processes of associated random variables. Using this Newman and Wright give conditions under which such partial sum processes converge to the Wiener process with both a one- and two-dimensional parameter. Here we discuss demimartingales for their own sake.

I. The Main Results

**Proposition 1.** A submartingale is a demisubmartingale.

**Proof.** Let $S_n$ be a submartingale and $f$ a nonnegative, nondecreasing
function of \( n \) variables. Then
\[
E(f(S_1, \ldots, S_n)(S_{n+1}-S_n)) = E(E(f(S_1, \ldots, S_n) | S_1, \ldots, S_n)) \geq E(f(S_1, \ldots, S_n)S_n) - E(f(S_1, \ldots, S_n)S_n) = 0.
\]

It is easy to provide examples of demisubmartingales which are not submartingales and we will do so later in the paper. For ease of notation we will often use the variables \( X_{n+1} = S_{n+1} - S_n \). Hence, throughout the remainder of the paper \( S_n = \sum_{i=1}^{n} X_i \) with \( S_0 = 0 \) unless otherwise stated.

**Theorem 1.** Let \( S_n \) be a demisubmartingale. Set
\[
M_k = \max\{S_i : 0 < i \leq k\} \quad \text{and} \quad m_k = \min\{S_i : 0 < i \leq k\}.
\]
Then for any real number \( t \) we have

1) \( t \Pr(M_k > t) \leq \int_{A_k} S_k \, dP \)

2) \( t \Pr(m_k \leq t) \geq E_S - \int_{B_k} S_k \, dP \geq E_S - E|S_k| \)

where \( A_k = \{w : M_k(w) > t\} \) and \( B_k = \{w : m_k(w) > t\} \).

**Proof:** The idea of the proof hinges on the fact that \( M_k \) is a nondecreasing function of \( S_1, \ldots, S_k \) and so the indicator function \( I(A_k) \) of \( A_k \) is a nonnegative, nondecreasing function of \( S_1, \ldots, S_k \).

Let \( i \) be the smallest index so that \( S_i > t \), i.e. \( i(w) = n \) if \( S_k(w) \leq t \) for all \( k < n \) and \( S_n(w) > t \). Then
\[
\int_{A_k} S_k \, dP = \sum_{n=1}^{k} \int_{\{i=n\}} S_k \, dP
\]
\[
= \sum_{n=1}^{k} \left( \int_{\{i=n\}} S_n \, dP + \int_{\{i=n\}} S_{k-n} \, dP \right)
\]
\[
= \sum_{n=1}^{k} \int_{\{i=n\}} S_n \, dP + \sum_{n=1}^{k-1} \int_{\{i=n\}} S_{k-n} \, dP
\]

Therefore, for any real number \( t \)
\[
\int A_k \leq \sum_{n=1}^{k} \int_{\{i=n\}} S_n \, dP + \sum_{n=1}^{k-1} \int_{\{i=n\}} S_{k-n} \, dP
\]

Hence,
\[
t \Pr(M_k > t) \leq \int_{A_k} S_k \, dP \]

and
\[
t \Pr(m_k \leq t) \geq E_S - \int_{B_k} S_k \, dP \geq E_S - E|S_k| \]
since $S_n$ is a demisubmartingale the last sum is nonnegative and (1) is proved.

To prove (2) take $i$ to be the smallest index so that $S_i \leq t$. Then by arguing as above, using $B'_k$ for the complement of $B_k$ in $\Omega$, it follows that

$$\int_{B'_k} S_k dP \leq t P(B_k') + \sum_{n=1}^{k-1} \int_{\Omega} I(B'_n) X_{n+1} dP.$$ 

Thus

$$t P(B_k') \geq \int_{B'_k} S_k dP - \sum_{n=1}^{k-1} \int_{\Omega} I(B'_n) X_{n+1} dP \geq \int_{B'_k} S_k dP - \sum_{n=1}^{k-1} \int_{\Omega} I(B'_n) X_{n+1} dP - \sum_{n=1}^{k-1} \int_{\Omega} I(B'_n) X_{n+1} dP$$

which completes the proof.

Let $[a,b]$ be a closed interval of real numbers. Define a sequence of stopping times $t_k(w)$ by

$$t_1(w) = \min\{j : S_j(w) \leq a\},$$

for even $k$, $t_k(w) = \min\{j : t_{k-1}(w) < j \leq m; S_j(w) \geq b\}$

for odd $k$, $t_k(w) = \min\{j : t_{k-1}(w) < j \leq m; S_j(w) \leq a\}$.

We make the convention here that the minimum of the empty set is $m$.

The number of upcrossings $U_m(w)$ of $[a,b]$ by $S_1(w), S_2(w), \ldots, S_m(w)$ is the number of times the sequence passes from below $a$ to above $b$. By $S^+_m(w)$ we mean the larger of $S_m(w)$ and $0$. 

$$\geq \int_{B'_k} S_k dP - E S_k + EX_1$$

$$\geq EX_1 - \int_{B_k} S_k dP$$
Theorem 2. Let $S_n$ be a demisubmartingale.

If $a < 0$, $E U_m \leq (b - a)^{-1} E S_m^+ + 1$.

If $a \geq 0$, $E U_m \leq (b - a)^{-1} (b + E S_m^+)$.

Proof. Consider as a special case $0 \leq a < b$ and a demisubmartingale $S_1, S_2, \ldots, S_m$ with $S_1 \equiv 0$. Replace $S_m$ with a new variable (still called $S_m$) with the new $S_m = S_m^+ + b$. We still have a demisubmartingale.

Define $Z = (S_{t_3} - S_{t_2}) + (S_{t_5} - S_{t_4}) + \ldots$

If $A_k = \{w: t_{2n}(w) \leq k \leq t_{2n+1}(w); \ n = 1, 2, \ldots\}$ then $I(A_k)$ is nondecreasing in the variables $S_1, \ldots, S_k$. (Here we use the fact that $S_1 \equiv 0$).

Thus

$$EZ = \sum_{k=2}^{m-1} \int_A X_{k+1} dP = \sum_{k=2}^{m-1} \int_{\Omega} I(A_k) X_{k+1} dP \geq 0.$$  

Now $S_m - S_1 = S_{t_1} - S_{t_1} = \sum_{n=1}^{m} (S_{t_{n+1}} - S_{t_n})$

$$= \Sigma^1(S_{t_{n+1}} - S_{t_n}) + \Sigma^2(S_{t_{n+1}} - S_{t_n})$$

where $\Sigma^1$ adds the terms with n odd and $\Sigma^2$ adds the terms with n even. We know that

$$EZ = E(\Sigma^2(S_{t_{n+1}} - S_{t_n})) \geq 0 \text{ and}$$

$$\Sigma^1(S_{t_{n+1}} - S_{t_n}) \geq U_m(b - a)$$

Thus

$$E(S_m - S_1) = E(S_m^+ + b - 0)$$

$$= ES_m^+ + B$$

$$\geq (b - a) E U_m$$

proving the special case.

To eliminate the assumption that $S_1 \equiv 0$, consider the sequence
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\[ Y_1 = S_2, \quad Y_2 = S_3, \ldots, \quad Y_m = S_{m+1}. \]

Then the upcrossings \( \overline{U}_m \) of \([a,b]\) by \( Y_1, \ldots, Y_m \) satisfies

\[ \overline{U}_m(w) \leq U_{m+1}(w) \leq \overline{U}_m(w) + 1 \]

and

\[ EU_m \leq EU_{m+1} \leq (b - a)^{-1} (b + ES_{m+1}^+) = (b - a)^{-1} (b + EY_m^+). \]

To eliminate the assumption that \( a \geq 0 \), notice that if \( a < 0 \), \( S_1 - a, S_2, S_3, \ldots, S_m \) still forms a demisubmartingale with possibly one more upcrossing over \([0,b-a]\) than \( S_1, S_2, \ldots, S_m \) makes over \([a,b]\). Finally, we mention that changing \( S_m \) to \( S_m + b \) increases \( S_m \) so the number of upcrossings by \( S_1, S_2, \ldots, S_{m-1}, S_m \) is less than or equal to the number of upcrossings by \( S_1, S_2, \ldots, S_{m-1}, S_m^+ + b \). This completes the proof.

**Theorem 3.** Let \( S_n \) be a demisubmartingale. If \( \limsup E|S_n| < \infty \), then there exists a random variable \( X \) such that \( E|X| < \infty \) and \( S_n \) converges to \( X \) almost surely.

**Corollary.** If, in addition to the hypotheses of Theorem 3,

\( [S_n : n = 1, 2, \ldots] \) is uniformly integrable, then \( S_n \) converges to \( X \) in the \( L^1 \) norm.

Once the upcrossing lemma (Theorem 2) is established, the proofs of Theorem 3 and its corollary are the same as the well known proofs of the corresponding results on submartingales. These results can be found for example in the book by Chung (1974, pp. 334-336).

It is easy to follow the example given by Doob in his book (Doob, 1953) to extend the results obtained so far to processes with a continuous parameter. We let \( \{S(t) : t \in [0,T]\}, 0 < T < \infty, \) be a stochastic process in \( L^1(\Omega,F,P) \). We say that \( S(t) \) is a demisubmartingale if for any \( k \) and \( \{t_n : n = 0,1,\ldots,k\} \subset [0,T] \) with \( 0 = t_0 < t_1 < \ldots < t_k = T \) then \( \{S(t_n) : n = 0,1,2,\ldots\} \) is a demisubmartingale.
Proposition 2. Let $S(t)$ be a demisubmartingale. If $a < b$ then $E S(a) < ES(b)$.

Proof. Take $0 = t_0$, $\alpha = t_1$, $\beta = t_2$, and $\gamma = T$. By assumption

$\{S(t_0), S(t_1), S(t_2), S(t_3)\}$ is a demisubmartingale. Since $f = 1$ is nonnegative and nondecreasing we have

$$\int O S(b) - S(a) dP = \int O f \cdot (S(t_2) - S(t_1)) dP \geq 0.$$

We refer the reader to Doob's book for the definition of a separable stochastic process and the fact that every stochastic process with a linear parameter has a separable version.

Theorem 5. Suppose $S(t)$ is a separable demisubmartingale. Denote by $A$ the set $\{w: \sup_{t} \{S(t,w): 0 \leq t \leq T < \infty \} > \lambda \}$ and by $B$ the set $\{w: \inf_{t} \{S(t,w): 0 \leq t \leq T < \infty \} \leq \lambda \}$. Then for any real number $\lambda$

1) $\lambda P(A) \leq \int_A S(T) dP$

2) $\lambda P(B) \geq ES(0) - E|S(T)|$.

Corollary. If $\{S(t): 0 \leq t \leq T < \infty \}$ is a nonnegative separable demisubmartingale, then

1) for $\alpha = 1$, $E(\sup_{t} S^\alpha(t)) \leq \frac{e}{e-1} + \frac{e}{e-1} E(S(T)\log^+ S(T))$

2) for $\alpha > 1$, $E(\sup_{t} S^\alpha(t)) \leq \left(\frac{\alpha}{\alpha-1}\right) \alpha E S^\alpha(T)$.

The proof of the corollary is immediate from the Maximal Inequality (Theorem 5) and Theorem 3.4' (p. 317) of Doob's book.

The Upcrossing Lemma and Maximal Inequality combine in the usual way to show that the sample paths of a demisubmartingale are nicely behaved.

Theorem 6. If $\{S(t): 0 \leq t \leq T < \infty \}$ is a separable demisubmartingale
then there is a set $D \in F$ with $P(D) = 0$ such that for all $\omega \not\in D$ the function $f(t) = S(t, \omega)$ defined for $t \in [0, T]$ is bounded and has no discontinuities of the second kind.

II. Examples

A. Let $X$ be any random variable in $L^1(\Omega, \mathcal{F}, P)$ and $c_n$ be any monotonically decreasing sequence of positive real numbers. If $EX = 0$ then $S_n = c_nX$ is a demimartingale since

$$E(f(S_1, \ldots, S_n)(S_{n+1} - S_n)) = E(f(S_1, \ldots, S_n)(c_{n+1} - c_n)X) \geq 0.$$  The easiest way to see that the last expectation is nonnegative is to use the fact that a random variable is always associated with itself (Esary, Proschan, Walkup, 1967) and the function $g(x) = cx$ for $c \geq 0$ is nondecreasing.

On the other hand $E(S_n|S_1, \ldots, S_m) = E(c_nX|X) = c_nX$ so $S_n$ is not a martingale unless $c_n$ is a constant sequence.

B. In the process of examining the relationship between demimartingales and weak martingales I came across an example of J. Berman (1976). It not only helped convince me that the demimartingale is a very different extension of martingales from that of weak martingales, but by a trivial modification of Berman's process we get a bounded demimartingale with countably many sample paths which is not a martingale. It is an example without the trivial nature of those in Example A.

Let $0 < \alpha < 1$ and $\Omega = \{f_0, f_1, \ldots\} \cup \{g_0, g_1, \ldots\}$ where for each $i \geq 0$, $f_i$ is the function on $0 \leq t < \infty$ defined by

$$f_i(t) = \begin{cases} 0 & , t < 2i \\ \frac{i}{\alpha} - \frac{t-i}{\alpha} & , t \geq 2i \end{cases}$$

and $g_i = -f_i$.  

Let $P$ be the probability function on $\Omega$ given by

$$p(f_i) = p(g_i) = 1/2(1-\alpha)^i, \ i \geq 0.$$
On the space $(\Omega, p)$, let \( \{X_n : n = 0, 1, 2, \ldots\} \) be the sequence of coordinate random variables:

\[
X_n(f_i) = f_i(n), \quad X_n(g_i) = g_i(n); \quad i \geq 0, \quad n \geq 0.
\]

Clearly \( |X_n| \leq 1 \) for all \( n \geq 0 \). We show that \( S_n = \sum_{i=0}^{\infty} X_i \) is a demi-martingale. The process is obviously not a martingale.

Consider any nondecreasing real function \( h \) of \( n \) variables.

Then

\[
E(h(X_1, \ldots, X_k) X_{k+1}) = 1/2 \sum_{i>0} \{ h(f_i(1), \ldots, f_i(k)) - h(g_i(1), \ldots, g_i(k)) \} f_i(k+1)p(f_i).
\]

By the definition of \( f_i(t) \) the summation above has only a finite number of nonzero terms. Since \( f_i(t) \geq 0, \quad p(f_i) \geq 0, \quad f_i(t) \geq g_i(t) \), and \( h \) is nondecreasing, the expectation in (1) must be nonnegative.

C. This example displays the purpose in developing a theory of demimartingales. It is due to Newman and Wright (1982). We start with the lattice \( \mathbb{Z}^2 \) of all order pairs \( (i, j) \) of integers. \( \{X(i,j) : (i,j) \in \mathbb{Z}^2\} \) is a collection of mean zero, finite variance associated random variables. Set

\[
S(n,k) = \sum_{i=1}^{n} \sum_{j=1}^{k} X(i,j)
\]

and for fixed \( m \),

\[
S_k = \max\{S(n,k) : 1 \leq n \leq m\}.
\]

Newman and Wright prove that \( S_k \) is a demisubmartingale. This fact along with the maximal inequality (Theorem 1) is the key to proving a maximal inequality for the two-parameter process \( S(n,k) \). This leads to the theorem of convergence to the Wiener process mentioned in the introduction.

D. In this example we follow the work of Professor S.D. Chatterji (1979). If \( \{\xi_{s,t}\} \), \( (s,t) \in D \subset \mathbb{R}^2 \), is some real valued stochastic pro-
cess on some probability space \((\Omega, \Sigma, P)\) parameterized by points of a subset of the plane, then clearly,

\[ X_s(w) = \{ t \mapsto \xi_{s,t}(w), \ t \in D_s \} \]

(\text{where } D_s = \{ t \in \mathbb{R} | (s, t) \in D \}) defines a vector-valued process \( \{X_s\} \), \( s \in I \), where \( I \) is the projection of \( D \) on the \( s \)-axis and the values of \( X_s \) are in some space \( E_s \) of functions on \( D_s \).

Assume that the process \( \xi_{s,t} \) has mean zero, associated increments in the sense that for any finite collection of points \((i, j) \in D, i \leq s, j \leq t, \xi_{s,t} \) can be written as a double sum of the type considered in Example C. If we further assume \( \xi_{s,t} \in L^2 \) then by theorem 6, the vector-valued process \( X_s \) has a version with regular sample paths.

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