§ 13. Preservation of Certain Filtrations

Suppose that $R$ is a ring, and that $\Lambda$ is an additive semigroup. We will say that $R$ is a $\Lambda$-graded ring (or simply a graded ring) if for each $\lambda \in \Lambda$, we have an additive subgroup $R_{\lambda}$ of $R$ such that

1) $R_{\lambda} R \subseteq R_{\lambda + \mu}$ ($\lambda, \mu \in \Lambda$) and

2) $R = \sum_{\lambda \in \Lambda} R_{\lambda}$.

Suppose again that $R$ is a ring and that $\Lambda$ is an additive semigroup. Suppose also that $\Lambda$ is partially ordered by a relation $<$ such that if $\lambda_1 < \lambda_2$, then $\lambda_1 + \mu < \lambda_2 + \mu$ ($\lambda_1, \lambda_2, \mu \in \Lambda$). We say that $R$ is a $\Lambda$-filtered ring if we have additive subgroups $R_{\lambda}$ of $R$ such that

1) $R_{\lambda} R_{\mu} \subseteq R_{\lambda + \mu}$ ($\lambda, \mu \in \Lambda$),

2) $R_{\lambda} \subseteq R_{\mu}$ if $\lambda < \mu$, and

3) $\bigcup_{\lambda \in \Lambda} R_{\lambda} = R$.

If $R$ is graded by the subspaces $R_{\lambda}$ ($\lambda \in \Lambda$), we obtain a filtered ring by means of the subspaces $R^\lambda = \sum_{\mu \leq \lambda} R_{\mu}$ of $R$. Similarly, if $R$ is a filtered ring, we obtain a graded ring $G(R)$ in the following way.

Let $G(R) = \sum_{\lambda \in \Lambda} \otimes G(R)_{\lambda}$, where $G(R)_{\lambda} = R^\lambda / R_{\lambda} R_{\mu}$ if $b \in G(R)_{\lambda}$ and $c \in G(R)_{\mu}$, and if $b' \in R^\lambda$ and $c' \in R^\mu$ map onto $b$ and $c$ respectively under the canonical maps $R^\lambda \to G(R)_{\lambda}$, $R^\mu \to G(R)_{\mu}$, then $bc$ is the image of $b'c'$ under the canonical map $R^{\lambda + \mu} \to G(R)_{\lambda + \mu}$.
If \( R \) is a ring and \( M \) is an \( R \)-module, we will say that \( M \) is an \( A \)-graded \( R \)-module if we have \( R \)-submodules \( M_\lambda \) \((\lambda \in A)\) such that
\[
M = \bigoplus_{\lambda \in A} M_\lambda.
\]
Similarly, we will say that \( M \) is a \( A \)-filtered \( R \)-module if we have \( R \)-submodules \( M_\lambda \) \((\lambda \in A)\) such that \( M_\lambda \subseteq M_\mu \) if \( \lambda < \mu \) and
\[
M = \bigcup_{\lambda \in A} M_\lambda. \quad \text{Given a graded \( R \)-module } M = \bigoplus_{\lambda \in A} M_\lambda, \text{ we obtain a filtered } R \text{-module by setting } M_\lambda = \bigoplus_{\mu < \lambda} M_\mu; \text{ and given a filtered } R \text{-module } M,
\]
we obtain a graded \( R \)-module \( G(M) \) by defining \( G(M)_\lambda = M_\lambda \bigcup_{\mu < \lambda} M_\mu \) and setting \( G(M) = \bigoplus_{\lambda \in A} G(M)_\lambda \).

Assume furthermore that \( A \) is totally ordered and that each \( \lambda \in A \) has at most finitely many distinct predecessors — i.e., that \( A \) is order isomorphic to the natural numbers. Then if \( R \) is a filtered ring and \( r \in R \), we may define the leading term of \( r \) to be the image \( \hat{r} \) of \( r \) in \( G(R)_A \), where \( \lambda \in A \) is the unique element such that \( r \in R_\lambda \) but \( r \notin R_\mu \) if \( \mu < \lambda \). If \( M \) is a filtered \( R \)-module, we define the notion of the leading term of elements of \( M \) analogously.

Now let \( A = C_\infty^\# \) be the semi-lattice generated by \( E_0(P,J,A) \).

If \( \lambda = \sum m_\lambda a_\lambda \in C_\infty \), we define the level \( |\lambda| \) of \( \lambda \) to be \( \sum m_\lambda \). Let \( < \) be the usual lexicographic order on \( C_\infty \) determined by the roots \( a_1, \ldots, a_k \). We now define a new order \( \prec \) on \( C_\infty \) as follows: if \( \lambda, \mu \in C_\infty \), we say that \( \lambda \prec \mu \) if \( |\lambda| < |\mu| \) or if \( |\lambda| = |\mu| \) and \( \lambda < \mu \). Clearly, \( \prec \) is a total order on \( C_\infty \). Also, the ordered semi-lattice \((A, \prec)\) has all the properties mentioned above: 1) if \( \lambda_1, \lambda_2, \mu \in A \) and \( \lambda_1 \prec \lambda_2 \), then...
\[ \lambda_1 + \mu < \lambda_2 + \mu; \text{ (2)} \text{ is a total order on } \Lambda; \text{ and (3)} \text{ each } \lambda \in \Lambda \text{ has at most finitely many predecessors. (Hence, } \Lambda \text{ is order isomorphic to the natural numbers; but if } \dim(O) > 1, \text{ it is not isomorphic to the natural numbers as an ordered semigroup.) } \]

The ring \( R_\mathbb{Z} \) has the structure of a graded ring, indexed by \( \Lambda \), as follows: \( R_\mathbb{Z} = \bigoplus_{\mu \in \Lambda} R_\mathbb{Z}_\mu \), where \( R_\mathbb{Z}_\mu \) is the set of \( f \in R_\mathbb{Z} \) such that \( \rho(a) = \exp(u(a)) \) (\( a \in A \)). Clearly, \( R_\mathbb{Z}_{\mu_1} R_\mathbb{Z}_{\mu_2} \subseteq R_\mathbb{Z}_{\mu_1 + \mu_2} \); so this indeed is a graded ring structure on \( R_\mathbb{Z} \). We note that the set of monomials \( \prod_{\beta \in \Lambda} \mathbb{R}^\beta \) for which \( \prod_{\beta \in \Lambda} |C_\beta| = \nu \) forms a basis for the subspace \( R_\mathbb{Z}_\mu (\mu \in \Lambda) \). Define \( R_\mathbb{Z}^\mu = \bigoplus_{\nu < \mu} R_\mathbb{Z}_\nu ^\mu \). Then the subspaces \( R_\mathbb{Z}_\mu ^\mu (\mu \in \Lambda) \) define a filtered ring structure on \( R_\mathbb{Z} \) with index set \( \Lambda \). Similarly, the subspaces \( M \otimes R_\mathbb{Z}_\mu ^\mu \otimes \mathbb{Q}[v] (\mu \in \Lambda) \) define a filtered ring structure (and also a filtered \( M \otimes \mathbb{Q}[v] \)-module structure) on the space \( M \otimes R_\mathbb{Z} ^\mu \otimes \mathbb{Q}[v] \).

As usual, we denote by \( Z_B \) the element \( 1/2(X_B + \theta(X_B)) \) (\( B \in \mathcal{P}_+ \)). If \( Z \in R_\mathbb{Z} \), \( B(Z, Z_B) = B(Z, Z_B) \); so \( R_\mathbb{Z} \cap (\bigoplus_{B \in \mathcal{P}_+} Z_B ^\lambda = (R_\mathbb{Z} ^\lambda c \otimes \mathbb{Q}[v] \). Hence by the Poincare-Birkhoff-Witt Theorem, the monomials

\[ (13.1) \quad Z_B ^{n_1} Z_{B_2} ^{n_2} \ldots Z_{B_s} ^{n_s} \quad (n_i > 0) \]

form a basis for \( r_\mathbb{Z} ^\lambda \) as a free right \( r_\mathbb{Z} ^\mathbb{M} \)-module (where \( \mathcal{P}_+ = \{ b_1, \ldots, b_s \} \) and \( B_1 < B_2 < \ldots < B_s \)). If \( \lambda \in \Lambda \), we let \( r_\mathbb{Z} ^\lambda \) denote the free right \( r_\mathbb{Z} ^\mathbb{M} \)-module spanned by the monomials (13.1) for which \( \sum_{i=1}^s n_i b_i \not\in \mathbb{Z} \lambda \).

Proposition 13.1. The subspaces \( r_\mathbb{Z} ^\lambda (\lambda \in \Lambda) \) define a filtered ring structure on \( r_\mathbb{Z} \). The associated graded ring is isomorphic to \( \mathcal{N}_\mathbb{Z} ^\mathbb{M} \).
Proof. For the first statement, we show that $\mathcal{H}^{\lambda \mu} \subseteq \mathcal{H}^{\lambda \mu}$ ($\lambda, \mu \in \Lambda$).

First we show that $\mathcal{H}^{\lambda}$ contains every element of the form $Z_{\gamma_1} \ldots Z_{\gamma_r}$ with $\gamma_1, \ldots, \gamma_r \in P_+$ and $\sum_{i=1}^{r} \gamma_i \in \mathcal{C} \lambda$. We prove this by induction on $r$. It is clearly true if $r = 1$, so assume that it is true for smaller values of $r$. Then $Z_{\gamma_2} \ldots Z_{\gamma_r}$ may be written as a right $\mathcal{H}^{\lambda}$-linear combination of the "standard" basis elements (13.1) in $\mathcal{H}^{\lambda'}$ ($\lambda' = (\gamma_2 + \ldots + \gamma_r) \cdot (\gamma_1)$). Hence we may as well assume that $Z_{\gamma_2} \ldots Z_{\gamma_r}$ has the form $\mathcal{H}^{\lambda}$ with

$$\sum_{i=1}^{r} \gamma_i \in \mathcal{C} \lambda - \gamma_1 | \mathcal{C}_2$$. We now proceed by induction on $\gamma_1$. Assume that $n_1 = \ldots = n_{i-1} = 0, n_i > 0$. If $\gamma_1 \leq \delta_i$ (in particular if $\gamma_1 = \delta_i$), we are done. Suppose that $\gamma_1 > \delta_i$. Then

$$Z_{\gamma_1} Z_{\delta_1} \ldots Z_{\delta_s} = Z_{\delta_1} Z_{\gamma_1} + 1/4([X_{\gamma_1} X_{\delta_s}]) + 1/4([X_{\gamma_1} X_{\delta_s}]) + 1/3([X_{\gamma_1} X_{\delta_s}]) + 1/2([X_{\gamma_1} X_{\delta_s}])$$

is in $\mathcal{H}^{\lambda}$ by the induction hypothesis. Similarly,

$$H = 1/3([H_{\delta_1} H_{\delta_s}]) \in \mathcal{H}^{\lambda} \cap \mathcal{H}^{\lambda'}$$

and

$$(n_{i-1})^2 H \in \mathcal{H}^{\lambda} \cap \mathcal{H}^{\lambda'}$$

Finally, $Z_{\delta_1} Z_{\gamma_1} Z_{\delta_1} \ldots Z_{\delta_s}$ is in $\mathcal{H}^{\lambda}$ by induction on $\gamma_1$, etc.
Now suppose that \( a \in \mathfrak{I}(P,A) \). Clearly, \( \mathfrak{M}_M \) normalizes \( \sum_{\lambda \in A} \mathfrak{M}_M^{(\lambda)} \); so by the preceding paragraph, it is obvious that \( \mathfrak{M}_M^{(\lambda)} \) is invariant under left multiplication by elements of \( \mathfrak{M}_M \). But then it is clear (again by the above) that \( \mathfrak{M}_M^{(\lambda)} \mathfrak{M}^{(\lambda)} \subseteq \mathfrak{M}_M^{(\lambda)} \mathfrak{M}^{(\lambda)} \) (\( \lambda, \mu \in \Lambda \)), as claimed.

For the second statement of the proposition, recall that the associated graded ring of \( \mathfrak{M}_M^{(\lambda)} \) is by definition the ring \( G(\mathfrak{M}) = \bigoplus_{\lambda \in A} G(\mathfrak{M}_{\lambda}) \), where
\[
G(\mathfrak{M}_{\lambda}) = \mathfrak{M}_M^{(\lambda)}/\mathfrak{M}_M^{(\lambda)}' \cdot \mathfrak{M}_M^{(\lambda)}. \]
Recall also that if \( b \in \mathfrak{M}_M^{(\lambda)} \), the leading term of \( b \) is the image \( \overline{b} \) of \( b \) in \( G(\mathfrak{M}_{\lambda}) \), where \( b \in \mathfrak{M}_M^{(\lambda)} \) but \( b \notin \mathfrak{M}_M^{(\lambda)} \) if \( \lambda' \prec \lambda \).

Clearly, each \( G(\mathfrak{M}_{\lambda}) \) is a \( \mathfrak{M}_M^{(\lambda)} \)-module; and \( G(\mathfrak{M}) \) is generated as a ring by \( \{ \overline{z}_\beta | \beta \in P_+ \} \) and \( G(\mathfrak{M})_0 = \mathfrak{M}_M \). Also it is clear that the "standard monomials"
\[
\overline{z}_\beta \overline{z}_\gamma \ldots \overline{z}_\delta (n_i \geq 0) \text{ form a basis for } G(\mathfrak{M}) \text{ as a right } \mathfrak{M}_M^{(\lambda)} \text{-module. Hence,}
\]
if \( u_1, \ldots, u_p \) is a basis for \( \mathfrak{M}_M^{(\lambda)} \), the monomials \( \overline{z}_\beta n_1 \overline{z}_\gamma n_2 \overline{z}_\delta n_3 \ldots \overline{z}_\beta n_p \) (\( n_i \geq 0, m_j \geq 0 \)) form a basis for \( G(\mathfrak{M}) \) as a \( \mathbb{F} \)-vector space.

Now consider the subspace \( \mathcal{W} \) of \( G(\mathfrak{M}) \) spanned by \( \overline{z}_\beta (\beta \in P_+) \) and \( \overline{u}_j (j = 1, \ldots, p) \). Clearly, \( [\overline{u}_i, \overline{u}_j] = [\overline{u}_i, \overline{u}_j] \) and
\[
[\overline{u}_i \overline{z}_\beta] = [\overline{u}_i \overline{z}_\beta] (i, j = 1, \ldots, p, \beta \in P_+). \]
Also,
\[
\overline{z}_\gamma \overline{z}_\beta \overline{y}_1 \overline{y}_2 + 1/24 \{ x_1 x_2 [x_1 x_2 \overline{y}_1 \overline{y}_2] \} + 1/24 \{ x_1 x_2 [\overline{x}_1 \overline{x}_2 \overline{y}_1 \overline{y}_2] \}
\]
\[
= \overline{z}_\gamma \overline{z}_\beta \overline{y}_1 \overline{y}_2 + 1/24 \{ x_1 x_2 [x_1 x_2 \overline{y}_1 \overline{y}_2] \} \mod \bigoplus_{\lambda \in A} \mathfrak{M}_M^{(\lambda)}. \]

Hence, \( [\overline{z}_\gamma \overline{z}_\beta] = 1/24 \{ x_1 x_2 [x_1 x_2 \overline{y}_1 \overline{y}_2] \} \) (\( \gamma, \delta \in \Lambda \)). This
shows first that $\mathcal{W}$ is a Lie subalgebra of $G(\mathcal{L})$, and second that the linear map $\epsilon: \mathfrak{n} \otimes \mathfrak{g}_{M}$ (semi-direct product) $\rightarrow G(\mathcal{L})$ such that $c(x_{\beta}) = 2\mathbf{Z}_{\beta}$ ($\beta \in \Phi$) and $c(u_{1}^{\beta}) = 0_{1}$ ($\beta = 1, \ldots, p$) is a Lie algebra isomorphism of $\mathfrak{n} \otimes \mathfrak{g}_{M}$ onto $\mathcal{W}$. Hence it extends to an isomorphism $\psi(\mathcal{C}) : \psi(\mathfrak{n} \otimes \mathfrak{g}_{M}) = \mathfrak{g}_{\mathcal{M}} \rightarrow \mathcal{U}(\mathcal{W})$ of universal enveloping algebras.

But by the Poincare-Birkhoff-Witt Theorem and the statement at the end of the preceding paragraph, $\psi(\mathcal{W}) = \mathcal{G}(\mathcal{L})$.

Remark 1. The graded ring structure on $\mathfrak{g}_{\mathcal{M}}$ is defined by the subspaces $\mathfrak{g}_{\lambda} \mathfrak{g}_{\mathcal{M}}$, where

$$\mathfrak{g}_{\lambda} \mathfrak{g}_{\mathcal{M}} = \{ b \in \mathfrak{g} \mid b^a = \text{exp}(\log a)b \ (a \in A) \ (\lambda \in \Lambda) \}.$$  

Clearly, the monomials $x_{\delta_1}^{n_1} \cdots x_{\delta_s}^{n_s}$ for which $\sum_{i=1}^{s} n_i \delta_i \mid \sigma^\delta = \lambda$ form a basis for $\mathfrak{g}_{\lambda} \mathfrak{g}_{\mathcal{M}}$.

Corollary 13.2. The submodules $\mathfrak{g}_{\lambda} \otimes \mathfrak{g}_{\mathcal{M}}$ ($\lambda \in \Lambda$) of $\mathfrak{g}_{\lambda} \otimes \mathfrak{g}_{\mathcal{M}}$ define a filtration of $\mathfrak{g}_{\lambda} \otimes \mathfrak{g}_{\mathcal{M}}$ by finitely-generated free $\mathcal{M}$-modules. The associated graded module is isomorphic as right $\mathcal{M}$-module to $\mathfrak{g}_{\lambda} \otimes \mathcal{U}_{r}$.

Proof. The sequence

$$0 \rightarrow \bigcup_{\lambda} \mathfrak{g}_{\lambda} \mathfrak{g}_{\lambda} \rightarrow \mathfrak{g}_{\lambda} \rightarrow \mathcal{G}(\mathcal{L})_{\lambda} \rightarrow 0$$

is an exact sequence of finitely-generated free $\mathfrak{g}_{\lambda} \mathfrak{g}_{\mathcal{M}}$-modules. Hence the
sequence

\[ 0 \to \left( \bigcup_{\lambda \in \Lambda} \mathfrak{N}^\lambda \right) \otimes \mathcal{M}^\lambda \to \mathfrak{N}^\lambda \otimes \mathcal{M}^\lambda \to \mathfrak{N}(\mathfrak{N}^\lambda) \otimes \mathcal{M}^\lambda \to 0 \]

is also exact. Therefore, the associated graded module of \( \mathfrak{N}^\lambda \otimes \mathcal{M}^\lambda \)

is \( \sum_{\lambda} \otimes \mathfrak{N}(\mathfrak{N}^\lambda) \otimes \mathcal{M}^\lambda \).

But \( \mathfrak{N}^\lambda \otimes \mathcal{M}^\lambda \) is isomorphic to \( \mathfrak{N}(\mathfrak{N}^\lambda) \) via the inverse of the map \( \delta: \mathfrak{N} \otimes \mathfrak{N}^\lambda \to \mathfrak{N}(\mathfrak{N}^\lambda) \) such that \( \delta(b \otimes c) = b \otimes c \) \((b \in \mathfrak{N}, c \in \mathfrak{N}^\lambda)\).

(Note that we consider \( \mathfrak{N} \otimes \mathfrak{N}^\lambda \) to be a right \( \mathfrak{N} \)-module under the action \((b \otimes c_1)c_2 = b \otimes c_1c_2\); so \( \delta \) is then an \( \mathfrak{N} \)-module isomorphism).

Remark 2. Suppose that \( \phi \in \mathfrak{R} \mathfrak{N}^\lambda \). If \( \gamma \in \mathfrak{N} \), let \( u = \gamma \mathfrak{N} \mathfrak{R} \). Then \( q(\phi) \in \mathfrak{R} \mathfrak{N}^\lambda \). Also, \( q(\phi) \in \mathfrak{N} \mathfrak{R} \mathfrak{N}^\lambda \) if \( \lambda - u \in \Lambda \) and is 0 if not.

This follows from part 3) of Proposition 11.6 and the definition of the spaces \( \mathfrak{N} \mathfrak{R} \mathfrak{N}^\lambda \).

Proposition 13.3. The mapping \( F_j: \mathfrak{N} \otimes \mathcal{M}^\lambda \to \mathfrak{N} \otimes \mathfrak{N} \mathfrak{R} \mathfrak{N}^\lambda \mathfrak{N} \mathfrak{R} \mathfrak{N}^\lambda \mathfrak{N} \) is a homomorphism of filtered \( \mathfrak{N} \)-modules.

Proof. We show that if \( b \in \mathfrak{N}^\lambda \) for some \( \lambda \in \Lambda \), then \( F_j(b) \in \mathfrak{N} \otimes \mathfrak{N} \mathfrak{R} \mathfrak{N}^\lambda \mathfrak{N} \mathfrak{R} \mathfrak{N}^\lambda \mathfrak{N} \).

Clearly we may assume that \( b \) is of the form \( z_{y_1} \ldots z_{y_j} \) with \((y_1, \ldots, y_j) \in \mathfrak{N} \mathfrak{N}^\lambda \). We proceed by induction on \( j \).

If \( j = 1 \), \( b = z_{y} \) with \( y \in \mathfrak{N} \). But by Proposition 11.6,

\( F_j(z_{y} \mathfrak{R}) = \mathfrak{N}(z_{y} \mathfrak{R}) \) is an \( \mathfrak{N} \)-linear combination of polynomial functions of the form \( B(z_{y} v^\mathfrak{N}) \) with \( V \in \mathfrak{N} \otimes \mathfrak{N} \). Also, \( B(z_{y} v^\mathfrak{N}) = L/2B(x_{y} v^\mathfrak{N}) \) and
\[ B(X_{n}, y_{n}^{-1}a) = B(X_{n}, y_{n}^{-1}a) = \exp(y \log a) B(X_{n}, y_{n}) \]; so clearly,
\[ F_{\lambda}(v)(Z_{\lambda}) = \phi_{\lambda}(v) \in \mathcal{M} \otimes \mathcal{P}^\lambda \mathcal{R} / \mathcal{R} \] with \( \lambda = \gamma | C \).

We claim that \( \phi_{\lambda}(v) \in \mathcal{P}^\lambda \mathcal{R} / \mathcal{R} \) with \( \lambda = \gamma | C \). For let \( J(n) = \sum_{\mu \in \Lambda} J_{\mu}(n) \) and \( \phi_{\lambda}(Z|n) = \sum_{\mu \in \Lambda} \phi_{\lambda, \mu}(Z|n) \) be the decomposition of \( J(n) \) and
\[ \phi_{\lambda}(Z|n) \] (\( Z \in \mathcal{P} \)) into their homogeneous components according to the decomposition \( \mathcal{R} / \mathcal{R} = \sum_{\mu} \mathcal{R} / \mathcal{R}_{\mu} \). Let \( u_{1} \) and \( u_{2} \) be the largest elements in \( \Lambda \) such that \( u_{1} \neq 0 \) and \( \phi_{\lambda, u_{2}}(Z_{\mu}) \neq 0 \). Then \( \phi_{\lambda}(Z_{\mu})J = \phi_{\lambda, u_{2}}(Z_{\mu})J_{u_{1}} + \) terms of \( \lambda \)-degree less than \( u_{1} + u_{2} \). Also, by Remark 2, the non-zero homogeneous components of \( q(Z_{\mu})J \) have \( \lambda \)-degree at most equal to \( u_{1} + \gamma | C \). But \( q(Z_{\mu})J = \phi_{\lambda}(Z_{\mu})J \) and \( \phi_{\lambda, u_{2}}(Z_{\mu})J_{u_{1}} \neq 0 \); so
\[ u_{1} + u_{2} \leq u_{1} + \gamma | C \]. Hence \( u_{2} \leq \gamma | C \) - i.e., \( \phi_{\lambda}(Z_{\mu}) \in \mathcal{P}^\lambda \mathcal{R} / \mathcal{R} \) with \( \lambda = \gamma | C \), as claimed. Therefore, \( F_{\lambda}(v)(Z_{\mu}) \in \mathcal{M} \otimes \mathcal{P}^\lambda \mathcal{R} / \mathcal{R} \) (\( \lambda = \gamma | C \)).

Now assume that our assertion is true if \( b = Z_{\gamma_{1}} \cdots Z_{\gamma_{j}} \). Hence, since the submodules \( \mathcal{M} \otimes \mathcal{P}^\lambda \mathcal{R} / \mathcal{R} \) define a filtered ring structure on \( \mathcal{M} \otimes \mathcal{P}^\lambda \mathcal{R} / \mathcal{R} \), it is clear that \( F_{\lambda}(v)(Z_{\gamma_{1}} \cdots Z_{\gamma_{j}}) \in \mathcal{M} \otimes \mathcal{P}^\lambda \mathcal{R} / \mathcal{R} \) with \( \lambda = (\gamma_{1} + \cdots + \gamma_{j}) | C \).
Similarly, writing $F_j(v)(Z_{\gamma^2} \cdots Z_{\gamma^J})$ as the sum of its homogeneous components and using Remark 2, we see that $q(Z)F_j(v)(Z_{\gamma^2} \cdots Z_{\gamma^J}) \in \mathfrak{H}_\lambda \mathfrak{H}_\lambda$ with $\lambda = (\gamma^1 \cdots \gamma^J)/Q$. Hence the same is true of $F_j(v)(b)$, as required.