6. CHARACTERISTICS OF TEACHING MATHEMATICAL PROBLEM SOLVING IN CHINA

Analysis of a Lesson from the Perspective of Variation

INTRODUCTION

Research over the past two decades has alluded that there are national patterns of mathematics teaching (Stigler & Hiebert, 1999). For example, Tweed and Lehman (2002) pointed out that there are distinctive features between Eastern and Western classrooms. Researchers have also found that there is evidence that teaching methods have evolved differently in particular countries. China appears to have developed a teaching approach that is different from Japan and Korea, though all three of them are rooted in the Confucian Heritage Culture (Givvin, Hiebert, Jacobs, Hollingsworth, & Gallimore, 2005; Park, 2006).

Recently, there has been increasing interest in uncovering the enigma of Chinese students’ outstanding mathematics achievements in international studies (cf. OECD, 2010, 2013). Considerable research has focused on characteristics of mathematics education in China, and it is recognized that “teaching with variation” is a Chinese way of promoting effective mathematics learning (Gu, Huang, & Marton, 2004; Wong, 2014; Wong, Lam, Sun, & Chan, 2009). Teaching with variation has almost become the teaching routine for Chinese mathematics teachers (Marton, Runesson, & Tsui, 2004) and has been applied either consciously or intuitively for a long time in China (Li, Peng, & Song, 2011).

In addition, problem solving has long been a staple of school mathematics (Stanic & Kilpatrick, 1988). In China there is a long history of interest in integrating problem solving into school mathematics (Siu, 2004; Stanic & Kilpatrick, 1988), and this tradition extends to the present (Cai & Nie, 2007). The development of students’ abilities to solve problems has remained one of the fundamental goals in school mathematics over the years. Problem solving is a distinctive mathematics activity from other mathematics learning areas such as mathematical concepts, algorithms, and theorems.

Researchers have identified some characteristics of teaching mathematical concepts from the pedagogy of variation in China. For instance, research shows that students are provided a series of problems in which essential features of mathematical concepts are kept unchanged, but the nonessential features of mathematical concepts are changed (Li et al., 2011). Huang and Leung (2004) found that teaching with
variation helps learners acquire knowledge step-wise, progressively develop experience in problem solving, and form well-structured knowledge. However, it is not clear how problem solving is taught from the pedagogy of variation. Given the significance of problem solving in mathematics education, the lack of such studies will limit our understanding of the full picture of mathematics teaching in China. This chapter aims to fill in this gap and make such a contribution through analyzing a well-structured Chinese mathematics lesson on solving right triangles in ninth grade.

THEORETICAL FRAMEWORK

Gu (1994) stated that teaching with variation is an important method through which students can easily understand relevant mathematical concepts. Furthermore, it illustrates the essential features by using different forms of visual materials and sometimes highlighting the essence of a concept by changing the nonessential features. The aim of teaching with variation is to understand the essence of an object and to form a scientific concept by eliminating nonessential distractions. Based on a series of longitudinal mathematics teaching experiments in China, Gu (1994) systematically synthesized and analyzed the concepts of teaching with variation. He identified and illustrated the two forms of variation, namely “conceptual variation” and “procedural variation.” Conceptual variation aims at providing students with multiple perspectives and experiences of mathematical concepts. Procedural variation aims to provide a process for the formation of concepts step-by-step so that students’ experiences in solving problems are manifested by the richness of varying problems and the variety of transferring strategies (Gu et al., 2004).

In particular, Gu et al. (2004) identified the following three types of variation: (1) varying the conditions of a problem: extending the original problem by varying the conditions, changing the results, and generalization; (2) varying the processes of solving a problem: using different methods of solving a problem; and (3) varying the applications of a method: applying the same method to a group of similar problems. Likewise, Cai and Nie (2007) identified three types of variation problems in Chinese mathematics education practice: one problem with multiple solutions, multiple problems with one solution, and one problem with multiple changes.

More theoretically and fundamentally, the research from Marton and Pang (2006) and Marton and Tsui (2004) indicated the following points for the theory of variation: learning is a process in which learners develop a certain capability or a certain way of seeing or experiencing; in order to see something in a certain way, the learner must discern specific features of the object; and experiencing variation is essential for discernment and is thus significant for learning content. Marton et al. (2004) argued that it is important to attend to what varies and what is invariant in a learning situation.

Building on the ideas from Marton et al. (2004, 2006), Watson and Mason (2006) also argued that because some features of problems are invariant while others are changing, learners are able to see the general through the particular, to generalize,
and to experience the particular. As pointed out by Cai and Nie (2007), teaching with variation by presenting a series of interconnected problems could help students understand concepts and master problem-solving methods, thereby developing students’ knowledge of mathematics.

In addition, Watson and Mason (2006) saw generalization as sensing the possible variation in a relationship and saw abstraction as shifting from seeing relationships as specific to the situation to seeing them as potential properties of similar situations.

In this study, we are going to analyze a lesson with the three key components of the theory of variation: invariant, varied, and discernment (Marton et al., 2004, 2006) and the three types of variations (Gu et al., 2004), as well as using “generalization” as a lens to check students’ learning (Watson & Mason, 2006).

METHODS

The Considerations: Why This Study Chooses This Lesson

The data included in the current study is a videotaped lesson. The topic of the lesson is solving right triangles. It belongs to a chapter about trigonometric function of acute angles in ninth grade, which includes two sections. The first section presents the definitions of trigonometric functions of acute angles including sine, cosine, tangent, and the second section presents solving right triangles that are the main focus of the selected lesson. Prior to the lesson, students learned the Pythagorean theorem, the definition of trigonometric functions of acute angles, and the methods on how to find the side length or angles of right triangles.

The lesson chosen for this study is a typical Chinese lesson under the background of China’s current mathematics curriculum reform. It includes six typical phases of the national pattern of teaching of a Chinese lesson (Peng, 2009). First, a context is set so as to lead to the mathematical problem that is going to be discussed in the lesson. Second, the new mathematical knowledge is introduced, on which the students are expected to collaboratively engage in inquiry-based learning. Third, a generalization is made. Fourth, students practice to enhance the new knowledge. Fifth, the students reflect what they have learned from the lesson. Sixthly, homework is assigned. The lesson lasts a total of forty-five minutes. Another reason to choose this lesson is that this lesson not only covers important content such as trigonometry, geometry, and algebra, but it also includes problem solving.

Data: The Lesson

The lesson was taught by Miss Li, a prospective teacher. She designed the lesson under the guidance of her teaching mentor in her university. She was in the last year of her 4-year bachelor degree program when she taught the lesson. It was enacted in a multimedia classroom with a projector, computer, and mathematical teaching.
materials including set squares and protractors. Teacher’s lecturing, students’ inquiry learning, and self-study form the main methods of teaching and learning during the lesson. Figure 1 shows the mathematics classroom where the teacher was discussing the problem with students.

Figure 1. The teacher was discussing solutions with students

Corresponding to the six phases, this lesson includes the following activities:

Activity 1: Introduction of the problem. The lesson began with an open question that asked, “How could you apply the knowledge of solving right triangles to solve real life problems?” Next, the teacher presented a real life problem situation on how to find the height of a broken tree (Figure 2), accompanied with five sets of

Figure 2. A problem on how to find the length of a broken tree
different givens regarding the different components of the tree. This data consisted of the length between the treetop and tree root (4 meters), the angle between treetop (broken branch the tree) and the ground (37°), the length between the top of the broken tree (trunk) and the ground (3 meters), and the angle formed by the broken part and the upright part is 53 degrees. The question was: “Which set of data can be used to find the height of the broken tree?”

Activity 2: Analysis of the problem. The teacher guided students to analyze the problem and then introduced the topic of the lesson—“solving right triangles, the process of finding the unknown measurements by using given measurements in a right triangle.” This analysis transferred the real life problem into a rigorous pure mathematical problem on how to solve a right triangle. The teacher re-stated the question: “Which of the five sets of data can be used to find the height of the broken tree?” The students were required to think about this question carefully and individually. After a while they were grouped to analyze the five sets of data under the knowledge framework of solving a right triangle and explored how to find the length of the tree by using the knowledge of solving right triangles. Figure 3 shows the visual representations of the five sets of data in right triangles, which were from students’ group work. In the drawings 3a, 3b, and 3e, the three hypotenuses show students’ different attempts. In Figure 3a, when only the length between the treetop and tree root (4 meters) is given, it is impossible to find original height of the broken tree. Corresponding to the situation in a right triangle, it means that, given the measure of one side, it is impossible to find the missing measurements. In Figure 3b, when only the angle between treetop and the ground (37°) is given, it is also impossible to find the length of the broken tree. Corresponding to the situation in a right triangle, it means that, given the measure of one angle, it is impossible to find the missing measurements. In Figure 3c, when both the length between the treetop and tree root (4 meters) and the length between the top of the broken tree and the ground (3 meters) are given, the length of the broken tree can be found. And it is \( \sqrt{3^2 + 4^2} + 3 = 8 \), Given the measures of two of the three sides, the missing measurements can be found using the Pythagorean theorem. In Figure 3d, when both the distance between the treetop and tree root (4 meters) and the angle between treetop and the ground (37°) are given, the length of the broken tree can be found: \( \frac{4}{\cos 37°} + 4\tan 37° = 8 \). Corresponding to the situation in a right triangle, it means that, given the measure of one side and one of the other two angles, the missing measurements can be found. In Figure 3e, when both the angle of the broken tree (53°) and the angle between treetop and the ground (37°) are given, it is impossible to find the length of the broken tree. Corresponding to the situation in a right triangle, it means that, given the measurements of two angles, it is impossible to find the missing measurements.
Activity 3: Generalization of problem solving. Students and the teacher generalized the conclusion about how to solve right triangles (Figure 4) together. The teacher stated: “One of the most important applications of trigonometry is to ‘solve’ a right triangle. By now, you should know that every right triangle has five measurements: the lengths of its three sides and the measures of its two acute angles. Solving a right triangle means to find the unknown measurements when some of them are given. You can use trigonometric functions to solve a right triangle if relevant information is provided. Is there anybody who wants to summarize which information is needed in order to solve a right triangle?” Students answered “the length of one side and the measure of one acute angle, or the lengths of two sides. Namely speaking, if we know the values of three out of the five elements of the right triangle (except the right angle, and at least one side must be included), we can find the values of the remaining elements using trigonometric ratios.”

Activity 4: Application of learned knowledge to various situations related to the same problem. There are two examples of application of the problem. First is a pure mathematical problem on solving a right triangle as follows:

In $\triangle ABC$, $\angle C = 90^\circ$, if $a = \sqrt{6}$, $b = \sqrt{2}$, solve this right triangle.

The second example is based on the question posed by the teacher: “If the tree is not broken, how could you find the length of the tree?” Specifically, it states that Xiao Ming wants to know the length of a big tree, which grows vertically on the campus. He stands 10 meters away from a tree root, and the angle of elevation from his position...
to the tree top is 50° measured by goniometer. The distance between his eyes and the ground is 1.5 meters. The question is: can you find the height of the tree? Figure 4 illustrates the conditions of this problem. Students are divided into groups to discuss the problem and they reach an agreement that right triangles can be constructed in order to solve the problem. Students proposed some interesting solutions, which will be discussed in the next section on varying the methods of solving the problem.

Figure 4. Visual representation of the second applied problem

The teacher encouraged students to find multiple ways to solve it. Below are two examples of students' strategies.

Student 1: “I draw a diagram according to the problem (as shown in Figure 5). And I find that there is a segment AD when point A and D are connected, and there is a right triangle ADC. Naturally, using the knowledge of solving a right triangle I can solve the problem.”

In $\triangle ADC$, $\angle CAD = 50°$, $AD = BF = 10m$.

Since $CD = AD \times \tan 50° = 10m \times 1.192 = 11.92m$, $AB = DF = 1.5m$.

We get the height of tree: $CF = DF + CD = 1.5m + 11.92m = 13.42m$.

Student 2: “We can construct right triangles to solve the problem. Extend segment CA and intersect the extension line of FB at point E, and there will be a triangle CEF. Since the length of segment AB is known, the right triangle AEB can be easily solved, and then the right triangle CEF can be solved (See Figure 5). In this way, the length of the tree can be found.”

Here is the student’s solution:

In $\triangle ABE$, $AB = 1.5m$, $\angle AEB = 50°$, $EB = \frac{AB}{\tan 50°} = \frac{1.5}{1.192} = 1.258$

In $\triangle CEF$, $EF = EB + BF = 1.258 + 10 = 11.258$.

$CF = EF \cdot \tan 50° = 11.258 \times 1.192 = 13.419 = 13.42$
We get that the height of the tree is 13.42 m.

Students then were divided into groups to discuss the multiple ways. With the teacher’s guidance, the multiple ways are compared. And finally, the specific steps and key points on how to use the knowledge of solving a right triangle to solve applied problems are generalized.

Activity 5: Review the lesson and assign homework. The teacher summarized the main contents of the lesson with the students. It included: (1) understanding what it means to solve a right triangle; (2) knowing that to solve a right triangle, at least one of the following two conditions are necessary: the length of one side and the measure of one acute angle, or the lengths of two sides; (3) using three tools for solving a right triangle: the trigonometric functions of acute angles, the Pythagorean Theorem, and the knowledge that the sum of the angles of a triangle is 180° (or the two acute angles are complementary, namely they add up to 90°); (4) being able to construct mathematical models and to solve simple practical problems by using knowledge regarding solving a right triangle.

Next, the teacher assigned two different types of tasks to the students. The first one is a basic task: In a right triangle ABC, angle A is 90°, solve the right triangle in terms of the given conditions as shown below:

1. if $a = 30$ and $b = 20$;
2. if angle B is $72°$ and $b = 14$.

The second one is an application task: Measure the height of the flagpole on campus with another classmate.

FINDINGS

By analyzing the activities implemented in this lesson, we have the following findings.
Varying the Conditions and Contexts of the Problems

As we see from the introduction and analysis of the problem from activities 1 and 2, the conditions of the problem are varied. In varying the different given measurements of right triangles, students understood the minimum conditions for solving right triangles, and thus understand the nature of solving right triangles. The visual representation of the five sets of data in right triangles drawn by students showed that students’ mathematical reasoning contained higher order thinking skills and understanding of problem solving with close attention to angles between two segments and length between two points. By keeping the problem situation invariant while varying the given conditions and modeling with varying visual representations, the students could discern the object of learning. For different conditions in the case of the right triangle, the number of solutions could be zero, one, or multiple. Table 1 indicates the three components of the variations.

<table>
<thead>
<tr>
<th>Invariant</th>
<th>Varied</th>
<th>Discernment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Context: the broken tree trunk</td>
<td>The types of conditions: side or angle</td>
<td>In what conditions, a right triangle is solvable</td>
</tr>
<tr>
<td>The quantities: the measures of sides and angles</td>
<td>The number of conditions</td>
<td></td>
</tr>
</tbody>
</table>

From activity 4, we found that the teacher presented two examples by varying both the conditions and situations of the broken tree trunk problem. Table 2 shows the three components of the variations.

<table>
<thead>
<tr>
<th>Invariant</th>
<th>Varied</th>
<th>Discernment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Right triangles</td>
<td>The conditions: sides and angles</td>
<td>Solving right triangles</td>
</tr>
<tr>
<td>Contexts</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalization of Problem Solving

Generalization is the result or refinement of the discernment. In activity 3, by comparing what varied (different conditions provided by students) against what remains invariant (the same situation), the object of learning should be discerned. In a right triangle, different conditions (known sides or angles) resulted in different methods for solving the triangle (sometimes one solution, multiple solutions, or no
solution). The students and the teacher generalized ideas about solving any right triangles in Figure 6:

![Figure 6. The summary and generalization of solving right triangles](image)

**Table 3. Varying the methods of solving the problem**

<table>
<thead>
<tr>
<th>Invariant</th>
<th>Varied</th>
<th>Discernment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measures of sides or angles</td>
<td>Focused on different triangles by Constructing new triangles</td>
<td>The height of the tree</td>
</tr>
<tr>
<td>Tangent is used</td>
<td></td>
<td>The relationships between the quantities</td>
</tr>
</tbody>
</table>

**DISCUSSION**

**Varying the Conditions and Contexts of the Problems**

Varying the conditions of the given problem provides students with an opportunity to experience a way of mathematical thinking in which they investigate the cases from special to general, from which students can see and construct mathematical concepts (Watson & Mason, 2006). Varying the conditions of a given problem provides a systemic experience for students to understand why a problem could be
solved with one or multiple solutions or a problem could be unsolvable (Gu et al., 2004).

By varying the conditions of the problem, the situation of the problem is simplified, structured, and made more precise so that the students would be able to easily understand the problem situation. This provides the foundational work for their problem solving (Gu et al., 2004).

Varying the context of problems means to change the contexts of problems while the mathematical essence of the problems remains similar. Gu et al. (2004) suggested that, during the process of solving problems, reorganizing separate but interrelated learning tasks as a group can provide a platform for learners to make connections between some interrelated concepts. In this sense, students are able to develop their experience in problem solving through “one problem with multiple changes” (Cai & Nie, 2007).

In the studied lesson, the teacher guided students to understand how to solve right triangles by creating related problems within different situations (both mathematical and contextual situations) and to apply the knowledge and strategies in different contexts.

From the application activities and the exercise assignment, we argue that varying the context of a problem can provide a scaffold for learners to make connections between relevant mathematical ideas; therefore, variation can enhance students’ problem-solving ability. In this form of variation, it is the structure of the tasks as a whole that encourages mathematical sense making (Watson & Mason, 2006).

Generalization

In this study, the teacher guided students to summarize the general rules of solving right triangles. The actual solution will depend on the specific problem, but the three tools are always used: the trigonometric functions, the Pythagorean theorem, the theorem that the sum of the angles of a triangle is 180°. There is not necessarily a “right” way to solve a right triangle. One way that is usually “wrong,” however, is solving for an angle or side in the first step, approximating that measurement, and then using that approximation to finish solving the triangle. This approximation will lead to inaccurate answers. As we can see from above, the varied conditions of the problem increases the complexity and cognitive requirements of problem solving, which helps students to understand the nature of solving a right triangle. Watson and Mason (2006) argued that generalizations created by students can become tools for developing more sophisticated mathematics and are a significant component of their mathematical progress.

Varying the Methods of Solving the Problem

Gu et al. (2004) stated that students’ experience in solving problems is manifested by the variety of transferring strategies. In the studied lesson, the teacher adopted this
form of variation by varying the method of solving the problem in order to foster students’ problem solving ability.

The lesson was successful in the sense that students obtained good learning results with a high average score, 91 out of 100, on the test. The teacher designed the test to assess student understanding and application of solving right triangles. That showed they had reached the target learning objectives including basic knowledge and skills for solving a right triangle. The assessment teacher group in the school gave good evaluations, such as “the teaching activities aroused students’ thinking” and “discussing the problems together was very helpful for students understanding and solving the problems.”

Our data show that in teaching problem solving, this form of variation provides an illustration of the way in which multiple methods to approach the same task can promote deep understanding. In summary, this variation offers a structured approach to exposing underlying mathematical forms, which can enhance students’ conceptual understanding of a series of related concepts (Lai & Murray, 2012).

FURTHER CONSIDERATIONS AND SUGGESTIONS

Characteristics of Teaching Problem Solving from a Variation Theoretical Perspective

From the perspective of variation, during the instruction of mathematical concepts students are provided a series of problems in which essential features of mathematical concepts are kept unchanged while the nonessential features of mathematical concepts are changed (Li et al., 2011). By doing this students are provided with multiple perspectives and experiences of mathematical concepts (Gu et al., 2004). Lesh and Zawojewski (2007) argued that problem solving is a learning activity that is more complex than the learning of mathematical concepts. It requires the problem solver to interpret a situation mathematically. The interpretation usually involves progression through iterative cycles of describing, testing, and revising ideas as well as identifying, integrating, modifying, or refining sets of mathematical concepts drawn from various sources. Through varying the conditions, contexts, and methods of solving the problem, the essential features of problem solving are highlighted. Students experience the process of problem solving, thus deepening their understanding and enhancing their ability.

Theoretically teaching with variation makes sense to foster students’ learning and problem solving ability, but it lacks enough empirical studies to verify it (Cai & Nie, 2007). Though our current study is just a case-based study, the fine-grained analysis provides solid evidence to confirm, “students’ experience in solving problems is manifested by the richness of varying problems and the variety of transferring strategies” (Gu et al., 2004, p. 322). Therefore the variations of problems help students make meaningful connections. Furthermore, our study has added new
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knowledge in this aspect by identifying the three types of variation for teaching mathematical problem solving and varying the conditions, contexts, and methods of solving the problem.

Researchers caution that teaching with variation does not necessarily lead to the development of basic skills. Sometimes, it can even limit the opportunities for fostering students’ higher-order thinking skills. Thus, further research is needed to investigate ways to effectively teach with variation (Mok, Cai, & Fung, 2008).

Towards a Teaching of Problem Solving with a Balance of Content-Oriented and Contextualization Oriented Teaching

With an increasing emphasis being placed on the applications of mathematics in real-life situations, the priority of contextualization of problems in the interest of facilitating connections is generally recognized as the common trend in mathematics education in the West (Clarke, 2006; Sun, 2013). However, there are more mathematical problems in classrooms in Hong Kong, Japan, and Korea (high-achieving regions in mathematics), compared to counterparts in the West in TIMSS 1999 video study (Leung, 2005). This implies that emphasizing contextual problems in mathematics teaching alone does not necessarily lead to excellence in students’ learning. One alternative may be making a balance between mathematical and contextual problems in mathematics teaching. To this end, teaching with variation may help us to make such a balance. As demonstrated in this study, the teacher presented variation problems with both contextual and mathematical situations for students to explore. This practice of teaching with variation in China may provide insights for mathematics educators in other cultures to reflect effective mathematics teaching.

CONCLUSIONS

Teaching with variation has been widely practiced in Chinese mathematics classrooms and is a teaching routine for Chinese mathematics teachers. The lesson featured in this study is a typical mathematics lesson in terms of teaching process and the use of variation. This study provides a vivid and concrete description of how teaching with variation was carried out in one type of lesson: teaching problem solving. Two types of variations are identified in our research, which may contribute to better understanding of teaching with variation in China. However, it is not our intent to generalize the findings to other lessons on problem solving in China.

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