Chapter 7
A Non-local Model Arising in Chemotaxis

Abstract  The current chapter deals with the biological phenomenon of chemotaxis. In the first place, a parabolic-parabolic Keller–Segel system is considered which describes the movement of some cell population towards a chemo-attractant produced by the population itself. Next, this version of Keller–Segel system is reduced to a non-local parabolic problem for the concentration of chemo-attractant in the case the chemo-attractant diffuses much faster than the cell population. Using the variational structure of the derived non-local parabolic problem we obtain some appropriate a priori estimates permitting us to derive global-in-time solutions when the total cell population is below the threshold $8\pi$. It is also proven that the global-in-time solution converges to the unique steady state solution in the radial symmetric case. When the cell population exceeds the threshold $8\pi$ then all the radially symmetric solutions exhibit finite-time blow-up on the origin of the considered sphere, i.e. single-point blow-up occurs.

7.1 Derivation of the Non-local Model

*Chemotaxis* is the movement of a motile cell or organism, in a direction corresponding to a gradient of increasing or decreasing concentration of a particular substance. Out of the many mathematical models that have been proposed to deal with particular aspects of chemotaxis, the one introduced by Keller and Segel in 1970 (cf. [17]) has received particular attention. The so called Keller–Segel model consists of two equations, describing the evolution of the population density $u(x, t)$ of motile cells (or organisms), and the concentration $v(x, t)$ of a chemical attracting substance, in a bounded domain $\Omega \subset \mathbb{R}^N$, where $N = 2, 3$, and in a time interval $[0, T]$:

\begin{align}
\varepsilon u_t &= \nabla \cdot (D_1 \nabla u - \chi u \nabla v) \quad \text{in } \Omega \times (0, T), \\
\tau v_t &= \Delta v - av + u \quad \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} &= v = 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega,
\end{align}

where $\varepsilon, \tau$ are positive constants.

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More precisely the first equation describes the random (Brownian) diffusion of the population of cells \( u \), which is biased in the direction of a drift velocity, proportional to the gradient of the concentration of the chemo-attractant \( \nabla v \). The diffusion coefficient is denoted by \( D_1 > 0 \) and the proportionality coefficient of the drift (mobility parameter) is denoted by \( \chi > 0 \). According to the second equation, the chemo-attractant \( v \), which is directly emitted by the cells, diffuses with a diffusion coefficient \( D_2 = 1/\tau > 0 \) on the substrate, while is generated proportionally to the density of cells and at the same time is degraded with a rate equal to \( a/\tau \geq 0 \).

A natural boundary condition, since it guarantees the conservation of total mass, is the no-flux type condition for \( u \), namely the first condition of (7.3) where \( \nu \) stands for the outer unit normal vector at \( \partial \Omega \). As for \( v \), a Dirichlet type boundary condition is assumed. Note that the parabolic system (7.1)–(7.4) preserves the nonnegativity of the initial conditions, i.e. \( u, v \geq 0 \) for \( t > 0 \), which is also expected to be true for the physical problem. For simplicity, \( D_1, \chi \) are considered to be constant and under suitable scaling can be taken \( D_1 = \chi = 1 \).

In view of experimental facts, the degradation of the chemical \( v \) is rather small so it can be taken \( a = 0 \) and therefore we actually focus on the following system

\[
\begin{align*}
\epsilon u_t &= \nabla \cdot (D_1 \nabla u - \chi u \nabla v) \quad \text{in } \Omega \times (0, T), \\
\tau v_t &= \Delta v + u \quad \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} &= v = 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega. 
\end{align*}
\]

For \((\epsilon, \tau) = (1, 0)\) we have an interesting case of the above system, when actually the chemo-attractant diffuses much faster than cell population. In that case system (7.5)–(7.8) has been studied thoroughly in various research papers, see [4, 11, 21] to name a few of them. Additionally, the same limiting case for the following chemotaxis system

\[
\begin{align*}
\epsilon u_t &= \nabla \cdot (\nabla u - u \nabla v) \quad \Omega \times (0, T), \\
\tau v_t &= \Delta v + u - \frac{1}{|\Omega|} \int_{\Omega} u \quad \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial v} - u \frac{\partial v}{\partial v} &= v = 0 \quad \text{on } \partial \Omega \times (0, T), \\
\int_{\Omega} v(x, t)dx &= 0 \quad \text{for } t \in (0, T), \\
u|_{t=0} &= u_0(x) \geq 0, \quad v|_{t=0} = v_0(x) \quad \text{in } \Omega, 
\end{align*}
\]

has been investigated in [27, 29, 32].

On the other hand, the other limiting case \((\epsilon, \tau) = (0, 1)\), which was first considered by Wolansky [34] is not so thoroughly studied. The latter limit actually describes the situation when chemo-attractant diffuses much faster compared to the cell population. In that case (7.5) together with non-flux boundary condition (7.7) entail
\[ \nabla u - u \nabla v = 0 \text{ in } \Omega, \]

which actually gives

\[ u = \frac{\lambda e^v}{\int_\Omega e^v \, dx}, \quad (7.14) \]

where \( \lambda \) is the total (conserved) mass of the population \( u \), i.e.

\[ \lambda = \int_\Omega u(x, t) \, dx = \int_\Omega u_0(x) \, dx. \]

Next by substituting (7.14) into (7.6) we derive the following non-local parabolic equation

\[ v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v \, dx} \text{ in } \Omega \times (0, T), \quad (7.15) \]

for the chemo-attractant, which is also associated with initial and boundary conditions

\[ v = 0 \text{ on } \partial \Omega \times (0, T), \quad (7.16) \]
\[ v(x, 0) = v_0(x) \geq 0 \text{ in } \Omega. \quad (7.17) \]

In the current chapter we will investigate the long-time behavior of problem (7.15)-(7.17) in the two-dimensional case, i.e. when \( \Omega \subset \mathbb{R}^2 \) which is a natural setting for species raised in a cell-culture dish. We will mainly focus on its blow-up time behavior which is actually linked with the case where chemo-attractant’s concentration becomes quite high.

### 7.2 Mathematical Analysis

#### 7.2.1 Preliminaries

Since \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary \( \partial \Omega \), then the functional

\[ J_\lambda(v) = \frac{1}{2} \| \nabla v \|_2^2 - \lambda \log \left( \int_\Omega e^v \right) + \lambda (\log \lambda - 1), \]

is \( C^2 \) for any \( v \in H_0^1(\Omega) \), and we obtain the Trudinger–Moser inequality indicated by

\[ \inf_{v \in H_0^1(\Omega)} J_{8\pi}(v) > -\infty. \quad (7.18) \]
Equation (7.15) is actually the gradient flow of functional $\mathscr{J}$, i.e. there holds

$$v_t = -\delta \mathcal{J}_\lambda(v) \quad \text{in } X = H_0^1(\Omega),$$

(7.19)

where $\delta \mathcal{J}_\lambda$ stands for the functional derivative of $\mathcal{J}_\lambda$, hence

$$\frac{d}{dt} \mathcal{J}_\lambda(v(\cdot, t)) = -\|v(\cdot, t)\|_2^2.$$  

(7.20)

Furthermore the mapping

$$v \in X \mapsto \frac{e^v}{\int_\Omega e^v} \in X,$$

is Lipschitz continuous on each bounded set of $X$, and therefore, (7.15) is well-posed in $X$. In particular, given initial value $v_0 \in X$, there exists a unique semi-group solution $v = v(\cdot, t) \in X$ locally in time, and henceforth its maximum existence time is denoted by $T = T_{\max} \in (0, +\infty]$, which is estimated from below by $\|\nabla v_0\|_2$. Then, due to (7.18) and via the parabolic regularity we obtain for $\lambda < 8\pi$

$$T = +\infty \quad \text{and} \quad \sup_{t \geq 0} \|v(\cdot, t)\|_\infty < +\infty,$$

(7.21)

whereas

$$\lim_{t \uparrow T} \|v(\cdot, t)\|_\infty = +\infty,$$

(7.22)

if $T < +\infty$. In the latter case we can define the set

$$\mathcal{S} = \left\{ x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \quad \exists t_k \uparrow T, \quad \text{s.t. } v(x_k, t_k) \to +\infty \right\} \neq \emptyset,$$

which is called the blow-up set of $v$. The blow-up set $\mathcal{S}$ can be also defined in case where (7.22) is fulfilled for $T = \infty$.

Note also that

$$v(\cdot, t) \geq \inf_{\Omega} v_0,$$

by the comparison principle, and henceforth, $v_0 \geq 0$ is assumed without loss of generality.

The parabolic Brezis–Merle’s inequality [9, 35], on the other hand, is concerned with the linear parabolic equation

$$v_t = \Delta v + f(x, t) \quad \text{in } \Omega \times (0, T), \quad v|_{\partial\Omega} = 0, \quad v|_{t=0} = 0,$$

where $\Omega$ is again a two-dimensional bounded domain with smooth boundary $\partial\Omega$. Its global form assures that any $\delta > 0$ admits $p > 1$ and $C > 0$ such that
\[ \sup_{t \in (0, T)} \| f(\cdot, t) \|_1 < 4\pi - \delta \Rightarrow \sup_{t \in (0, T)} \| e^{v(\cdot, t)} \|_p \leq C. \]

This induces the local version, and given subdomains \( \omega \) and \( \hat{\omega} \) satisfying \( \omega \subset \subset \hat{\omega} \subset \subset \Omega \) and \( \delta > 0 \), we obtain \( p > 1 \) and \( C > 0 \) both determined by \( \sup_{t \in (0, T)} \| f(\cdot, t) \|_1 \) such that

\[ \sup_{t \in (0, T)} \| f(\cdot, t) \|_{L^1(\hat{\omega})} < 4\pi - \delta \Rightarrow \sup_{t \in (0, T)} \| e^{v(\cdot, t)} \|_{L^p(\omega)} \leq C. \]

**Remark 7.2.1** Here we should point out that local-existence of problem (7.15)–(7.17) can be also derived by using two iteration schemes with starting point an upper-lower solution pair \( (z, w) \) defined as

\[
\begin{align*}
\dot{z}_t &\leq \Delta z + \frac{\lambda e^z}{\int_{\Omega} e^z \, dx} \quad \text{in} \quad \Omega \times (0, T), \\
\dot{w}_t &\geq \Delta w + \frac{\lambda e^w}{\int_{\Omega} e^w \, dx} \quad \text{in} \quad \Omega \times (0, T), \\
z(x, t) &\leq 0 \leq w(x, t), \quad \text{on} \quad \partial\Omega \times (0, T), \\
z(x, 0) &\leq v_0(x) \leq w(x, 0) \quad \text{in} \quad \Omega,
\end{align*}
\]

and following a similar approach to Proposition 1.2.2.

### 7.2.2 Blow-Up Results

This subsection is devoted to the investigation of the blow-up behavior of problem (7.15)–(7.17). Our first result rules out the possibility of blow-up taking place on the boundary \( \partial\Omega \) in case of a convex domain \( \Omega \). In particular, the following holds

**Lemma 7.1** If \( T < +\infty \) and \( \Omega \) is convex, then \( \mathcal{S} \subset \Omega \).

**Proof** Let \( v = v(x, t) \) be a solution to (7.15) with \( v|_{t=0} = v_0(x) \). Having assumed \( v_0 \geq 0 \), we obtain \( v = v(x, t) \geq 0 \), and then

\[ \sup_{t \in [0, T]} \| v(\cdot, t) \| < +\infty, \]

follows for any \( p \geq 1 \) by virtue of the global parabolic Brezis–Merle inequality.

Notably \( v \) is a solution to

\[
\begin{align*}
\dot{v}_t &= \Delta v + \sigma(t)e^v \quad \text{in} \quad \Omega \times (0, T), \\
v|_{\partial\Omega} &= 0, \quad v|_{t=0} = v_0(x)
\end{align*}
\]
for $\sigma(t) = \frac{\lambda}{\int_{\Omega} e^{v(t)}}$ and the method of the moving plane, [8], is applicable. Using (7.27), now we can apply the argument of [5], and consequently, there is an open set $\hat{\omega}$ containing $\partial \Omega$ and a constant $C > 0$ such that

$$\sup_{t \in [0, T)} \|v(\cdot, t)\|_{L^\infty(\Omega \cap \hat{\omega})} \leq C,$$

and therefore, $\mathcal{S} \subset \Omega$. □

Remark 7.2.2 Due to a uniform $L^1$–estimate of the solution of (7.15) the result of the above Lemma can be extended to higher dimensions $N \geq 2$. In fact, multiplying the equation of (7.15) by the eigenfunction $\phi_1(x) > 0$ corresponding to the principal eigenvalue $\lambda_1$ of $-\Delta_P$ and integrating all over $\Omega$ we obtain

$$\frac{d}{dt} \int_{\Omega} v(x, t) \phi_1(x) \, dx \leq -\lambda_1 \int_{\Omega} v(x, t) \phi_1(x) \, dx + \lambda M, \quad 0 < t < T,$$

for $M = \max_{\Omega} \phi_1(x) > 0$, which implies

$$\int_{\Omega} v(x, t) \phi_1(x) \, dx \leq C := C(v_0, \lambda_1, \lambda, M) \quad \text{for any } 0 < t < T. \quad (7.28)$$

Since $\Omega$ is convex, using the method of moving planes, [8], we can find $\bar{\Omega}_0 \subset \Omega$ such that

$$\int_{\Omega} v(x, t) \, dx \leq \frac{k + 1}{m} \int_{\Omega_0} v(x, t) \phi_1(x) \, dx \leq \frac{k + 1}{m} \int_{\Omega} v(x, t) \phi_1(x) \, dx < C_1, \quad (7.29)$$

by (7.28) for any $0 < t < T$ and $m = \min_{\bar{\Omega}_0} \phi_1(x) > 0$. Using now estimate (7.29), which for $N = 2$ stems from the parabolic version of the Brezis–Merle’s inequality, along with the arguments introduced in [5] we derive the desired result. If $v = v(x, t)$ is radially symmetric and decreasing in $r = |x|$, and if (7.22) holds, then $\mathcal{S} = \{0\}$ by the same reasoning. We also note that in the case of $T_{\text{max}} = +\infty$, any radially symmetric solution becomes decreasing in $r = |x|$ eventually, see [25].

Note that stationary problem to (7.15)–(7.17) is described by

$$- \Delta v_* = \frac{\lambda e^{v_*}}{\int_{\Omega} e^{v_*}} \quad \text{in } \Omega, \quad v_* = 0 \quad \text{on } \partial \Omega. \quad (7.30)$$

Let now

$$E = \{v_* \mid v_* \text{is a classical solution to (7.30)}\}$$

then we have the following result due to Wolansky, see Theorem 8 in [34]. Since the local well-posedness in time of (7.15)–(7.17) in $X = H^1_0(\Omega)$ together with the Trudinger–Moser inequality are used then its proof is valid only to $N = 1, 2$. However, here a simpler proof is provided.
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Lemma 7.2 If \( E = \emptyset \), then it holds that

- \( (i) \)
  \[
  \lim_{t \uparrow T} \int_{\Omega} e^{v(\cdot, t)} = +\infty, \quad (7.31)
  \]

- \( (ii) \) \( S \neq \emptyset \), where the case \( T = +\infty \) is also permitted.

Proof We first prove statement \( (ii) \). If we assume that \( S = \emptyset \), then

\[
\sup_{t \in [0, T)} \| v(\cdot, t) \|_\infty < +\infty.
\]

The latter implies \( T = +\infty \) and \( E \neq \emptyset \) by a standard argument using Lyapunov function [10], which leads to a contradiction.

Next we proceed with the proof of \( (i) \). Therefore if we assume that

\[
\lim_{t \uparrow T} \int_{\Omega} e^{v(\cdot, t)} < +\infty,
\]

then we obtain

\[
\lim_{t \uparrow T} \mathcal{J}_\lambda(v(\cdot, t)) \geq \limsup_{t \uparrow T} \left\{ -\lambda \log \left( \int_{\Omega} e^{v(\cdot, t)} \right) \right\} + \lambda (\log \lambda - 1) > -\infty, \quad (7.32)
\]

and finally

\[
\frac{1}{2} \liminf_{t \uparrow T} \| \nabla v(\cdot, t) \|_2^2 \leq \mathcal{J}_\lambda(v_0) + \lambda \liminf_{t \uparrow T} \log \left( \int_{\Omega} e^{v(\cdot, t)} \right) - \lambda (\log \lambda + 1) < +\infty.
\]

The latter estimation guarantees \( T = +\infty \) owing to the well-posedness of (7.15)–(7.17) in \( X = H_0^1(\Omega) \). Furthermore, there are \( \delta \in (0, 1) \), \( t_k \uparrow +\infty \), and \( C > 0 \) such that \( t_{k+1} > t_k + 1 \) and

\[
\sup_{t \in (t_k, t_k + \delta)} \| \nabla v(\cdot, t) \|_2 \leq C.
\]

The latter implies

\[
\sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} \| v_t(\cdot, t) \|_2^2 \, dt \leq \int_0^{\infty} \| v_t(\cdot, t) \|_2^2 \, dt < +\infty,
\]
by (7.32) and (7.20), and therefore,

$$\lim_{k \to \infty} \int_{t_k}^{t_k + \delta} \| v(t) \|^2 dt = 0.$$ 

Then, we can find $t'_k \in (t_k, t_k + \delta)$ satisfying $\| v(t'_k) \|_2 \to 0$. Since $\| \nabla v(t'_k) \|^2 \leq C$, we obtain a subsequence, denoted by the same symbol, such that $v(t'_k) \to w$ weakly in $X$. This implies $e^v(t'_k) \to e^w$ strongly in $L^p(\Omega)$, $p \geq 1$, by the Trudinger–Moser inequality, and then $w \in E$ follows, which is a contradiction. \hfill \Box

The following result which is due to Wolansky, \cite{34}, actually says that above the threshold $8\pi$ a mass concentration occurs at the origin $x = 0$ in the case of the unit disc $\Omega = B_1(0) = \{ x \in \mathbb{R}^2 \mid 0 \leq |x| < 1 \}$.

**Theorem 7.1** If $\Omega = B_1(0)$, $v_0 = v_0(|x|)$ is smooth and $\lambda \geq 8\pi$, then

$$\frac{e^v}{\int_{\Omega} e^v} \to \delta_0(dx) \quad \text{as} \quad t \uparrow T, \quad (7.33)$$

for the solution $v$ of (7.15)–(7.17), where $T \leq +\infty$.

**Proof** Let $\Omega = B_1(0)$, $v_0 = v_0(|x|) \geq 0$, and $\lambda \geq 8\pi$. We obtain $E = \emptyset$ in this case, and therefore, $\mathcal{S} \neq \emptyset$ and also (7.31) holds true. Since $v = v(|x|, t)$, then any $x_0 \in \Omega \setminus \{ 0 \}$ admits $0 < r \ll 1$ such that

$$\sup_{t \in (0, T)} \left\| \frac{\lambda e^{v(t)}}{\int_{\Omega} e^{v(t)}} \right\|_{L^1(B_r(x_0))} < 4\pi.$$ 

Accordingly, the local parabolic Brezis–Merle’s inequality applied to problem (7.15)–(7.17) guarantees

$$\sup_{t \in (0, T)} \left\| e^{v(t)} \right\|_{L^p(B_r(x_0))} < +\infty,$$

with $p > 1$, and therefore, there holds that

$$\sup_{t \in (0, T)} \left\| \frac{\lambda e^{v(t)}}{\int_{\Omega} e^{v(t)}} \right\|_{L^p(B_r(x_0))} < +\infty,$$

since $v \geq 0$. The latter implies

$$\sup_{t \in (0, T)} \left\| v(t) \right\|_{W^{2,p}(B_{r/2}(x_0))} < +\infty,$$

via the local parabolic regularity, and hence $x_0 \notin \mathcal{S}$ by Sobolev’s imbedding theorem. Consequently, we have $\mathcal{S} = \{ 0 \}$ and so (7.33) is valid. \hfill \Box
Although the case $T = +\infty$ is also admitted in the above theorem, in [16] is proven the following more delicate result.

**Theorem 7.2** If $\Omega = B_1(0)$ and $v_0 = v_0(|x|)$ is smooth, then finite-time blow-up occurs for the solution of (7.15)–(7.17), i.e. $T < +\infty$, provided that $\lambda > 8\pi$.

**Proof** First note, that $v$ is radial symmetric, i.e. $v(x, t) = v(r, t)$ where $r = |x|$, since $\Omega = B(0, 1)$. Furthermore, for $u$ defined by (7.14) we have $\nabla u = u\nabla v$ and therefore via integration by parts,

$$-\int_{\Omega} u \nabla \cdot \psi \, dx = \int_{\Omega} u \psi \cdot \nabla v \, dx,$$  
(7.34)

for any $\psi \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$. In view of the following problem

$$v_t = \Delta v + u \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial \Omega,$$  
(7.35)

we can express $v$ as follows

$$v = (-\Delta_D)^{-1} u - (-\Delta_D)^{-1} v_t,$$

where $w = (-\Delta_D)^{-1} u$ denotes the solution of the following Dirichlet problem

$$-\Delta w = u \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \partial \Omega.$$

Next we define the function

$$\rho(x, x') = \psi(x) \cdot \nabla_x G(x, x') + \psi(x') \cdot \nabla_{x'} G(x, x'),$$

where $G = G(x, x')$ is the Green’s function of $-\Delta_D$ satisfying $G(x', x) = G(x, x')$ and

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|} + K(x, x'),$$  
(7.36)

with $K \in C^{2+\theta}_{loc}(\overline{\Omega} \times \Omega \cup \Omega \times \overline{\Omega})$ for $0 < \theta < 1$.

Thus, the right-hand side of (7.34), see [32], is equal to

$$\int_{\Omega} u \psi \cdot \nabla v \, dx = \int_{\Omega} u \psi \cdot \nabla \{(-\Delta_D)^{-1} u - (-\Delta_D)^{-1} v_t\} \, dx$$

$$= \frac{1}{2} \int_{\Omega \times \Omega} \rho(x, x') u(x, t) u(x', t) \, dx \, dx' - \int_{\Omega} u \psi \cdot \nabla(-\Delta_D)^{-1} v_t \, dx.$$  

Considering now a test function of the form $\psi(x) = x \varphi(|x|)$ for $\varphi = \varphi(|x|) \in C^\infty_0(\Omega)$ satisfying $0 \leq \varphi \leq 1$ and $\varphi = 1$ near $x = 0$ then, $\nabla \cdot \psi|_{x=0} = 2$ and therefore,

$$\int_{\Omega} u \nabla \cdot \psi \, dx = 2\lambda + o(1),$$  
(7.37)
holds as $t \uparrow T$ by (7.33).

The relation (7.36), on the other hand, guarantees

$$\rho(x, x') = -\frac{1}{2\pi} + L(x, x'),$$

with $L = L(x, x') \in C(\overline{\Omega} \times \overline{\Omega})$ satisfying $L(0, 0) = 0$, and therefore,

$$\frac{1}{2} \int \int_{\Omega \times \Omega} \rho(x, x') u(x, t) u(x', t) dx dx' = -\frac{\lambda^2}{4\pi} + o(1),$$

as $t \uparrow T$. Thus, by virtue of (7.34) and (7.37) we obtain

$$-\int_{\Omega} u(\psi \cdot \nabla)(-\Delta_D)^{-1} v_i \, dx = \frac{\lambda^2}{4\pi} - 2\lambda + o(1),$$

as $t \uparrow T$.

In addition, the relation $V = (-\Delta_D)^{-1} v_i$ implies $(rV_r)_r = -r v_i$, and hence

$$rV_r(r, t) = -\int_{0}^{r} s v_i(s, t) ds.$$

The latter implies

$$(\psi \cdot \nabla)(-\Delta_D)^{-1} v_i = \varphi(r) r \partial_r (-\Delta_D)^{-1} v_i = -\varphi(r) \int_{0}^{r} s v_i(s, t) ds$$

$$= -\frac{\varphi(r)}{2\pi} \int_{B(0, r)} v_i(s, t) \, dx,$$

and thus we derive

$$-\int_{\Omega} u(\psi \cdot \nabla)(-\Delta_D)^{-1} v_i \, dx \leq \frac{\lambda}{2\pi} \sup_{r} \varphi(r) \int_{B(0, r)} v_i \, dx \leq \frac{\lambda}{2\pi} \|v_i\|_1,$$

since $u \geq 0$ and

$$\|u\|_1 = \lambda.$$

Finally by virtue of (7.38) and taking also into account that $\lambda > 8\pi$ we end up with

$$\lim_{t \uparrow T} \inf \|v_i(\cdot, t)\|_1 \geq \frac{\lambda}{2} - 4\pi > 0,$$

which entails

$$\|v_i(\cdot, t)\|_2 \geq \delta,$$
with $\delta > 0$ independent of $t \geq 1$, in case of a global-in-time solution, i.e. for $T = +\infty$.

Let now $0 < \mu_1 < \mu_2 \leq \ldots$ be the eigenvalues of $-\Delta_D$ with corresponding $L^2$-normalized eigenfunctions $\varphi_j = \varphi_j(x)$, $j = 1, 2, \ldots$, i.e.

$$-\Delta \varphi_j = \mu_j \varphi_j \text{ in } \Omega, \quad \varphi_j = 0 \text{ on } \partial \Omega,$$

with $\|\varphi_j\|_2 = 1$.

Then, we have the following asymptotic behavior, [30],

$$\mu_j \sim j \quad \text{and} \quad \|\varphi_j\|_\infty \leq C j^{1/4} \text{ as } j \to \infty. \quad (7.42)$$

If $T = +\infty$, then by virtue of (7.27) for $p = 2$, we can find $t_k \to +\infty$ and a $v_\ast \in L^2(\Omega)$ such that

$$v(\cdot, t_k) \rightharpoonup v_\ast(\cdot) \text{ weakly in } L^2(\Omega) \text{ as } k \to \infty. \quad (7.43)$$

Set $g_j(t) = \langle v(\cdot, t), \varphi_j \rangle$ then there holds

$$\dot{g}_j = -\mu_j g_j + \langle u, \varphi_j \rangle,$$

and thus

$$g_j(t + t_k) = e^{-\mu_j t} g_j(t_k) + \int_0^t e^{-(t-s)\mu_j} \langle u(\cdot, s + t_k), \varphi_j \rangle ds.$$

The above relation leads, taking also into account (7.39), to the estimate

$$\left| \langle v(\cdot, t + t_k), \varphi_j \rangle \right| \leq e^{-\mu_j t} \|v(\cdot, t_k)\|_2 + \lambda \|\varphi_j\|_\infty \mu_j^{-1},$$

which by virtue of (7.27) and (7.42) implies

$$\left| \langle v(\cdot, t + t_k), \varphi_j \rangle \right| \leq A_j,$$

for $t \geq 1$ and $k = 1, 2, \ldots$ where $A_j = C(e^{-\alpha j} + \lambda j^{-2}) > 0$ for $j \gg 1$, $\alpha > 0$, and $C > 0$, satisfying $\sum_{j=1}^\infty A_j^2 < +\infty$.

Now using (7.33) and (7.43) we derive

$$g_j(t + t_k) \to e^{-\mu_j t} \langle v_\ast, \varphi_j \rangle + \int_0^t e^{-(t-s)\mu_j} \lambda \varphi_j(0) ds$$

$$= e^{-\mu_j t} \langle v_\ast, \varphi \rangle + \frac{\lambda \varphi_j(0)}{\mu_j} (1 - e^{-\mu_j t}),$$

as $k \to \infty$ uniformly in $t \geq 1$ for each $j$, and therefore,

$$v(\cdot, t + t_k) \to V(\cdot, t) \quad \text{as } k \to \infty,$$
in \( L^2(\Omega) \) uniformly in \( t \geq 1 \), where

\[
V(\cdot, t) = \sum_{j=1}^{\infty} \left\{ e^{-\mu_j t} \left( v_*, \varphi_j \right) + \frac{\lambda \varphi_j(0)}{\mu_j} (1 - e^{-\mu_j t}) \right\} \varphi_j.
\]

Similarly,

\[
v_t(\cdot, t + t_k) \to W(\cdot, t) \quad \text{as} \quad k \to \infty,
\]

in \( L^2(\Omega) \) uniformly in \( t \geq 1 \), where

\[
W(\cdot, t) = \sum_{j=1}^{\infty} e^{-\mu_j t} \left\{ -\mu_j \left( v_*, \varphi_j \right) + \lambda \varphi_j(0) \right\} \varphi_j = V_t,
\]

and thus

\[
\| W(\cdot, t) \|^2_2 \geq \delta^2,
\]

by virtue of (7.41).

On the other hand, (7.44) implies

\[
\| W(\cdot, t) \|^2_2 = \sum_{j=1}^{\infty} e^{-2\mu_j t} \left| -\mu_j \left( v_*, \varphi_j \right) + \lambda \varphi_j(0) \right|^2 \to 0 \quad \text{as} \quad t \to +\infty,
\]

which actually contradicts (7.45). This completes the proof of the theorem.

The above theorem entails that the formation of collapse with dis-quantized mass occurs in finite time for the limiting case \( \varepsilon = 0 \), in contrast with what happens in other limiting case \( \tau = 0 \), regarding the problem (7.5)–(7.8).

### 7.3 An Associated Competition-Diffusion System

In the current subsection a brief investigation of a non-local reaction-diffusion system stems from (7.15)–(7.17) is delivered. Specifically, we consider the following

\[
z_t = \Delta z + f(z, w) \quad \text{in} \quad \Omega \times (0, T), \quad (7.46)
\]
\[
w_t = \Delta w + g(z, w) \quad \text{in} \quad \Omega \times (0, T), \quad (7.47)
\]
\[
z(x, t) = w(x, t) = 0, \quad \text{on} \quad \partial \Omega \times (0, T), \quad (7.48)
\]
\[
z(x, 0) = z_0(x), \quad w(x, 0) = w_0(x), \quad \text{in} \quad \Omega, \quad (7.49)
\]
where

\[ f(z, w) = \frac{\lambda e^z}{\int_{\Omega} e^w \, dx} \quad \text{and} \quad g(z, w) = \frac{\lambda e^w}{\int_{\Omega} e^z \, dx}. \]

Notably, in case where \( z(x, t) = v(x, t) \) then system (7.23)–(7.26) is reduced to problem (7.15)–(7.17).

Next, we observe that

\[ \frac{\partial f(z, w)}{\partial w} \leq 0 \quad \text{and} \quad \frac{\partial g(z, w)}{\partial z} \leq 0, \quad (7.50) \]

and thus (7.23)–(7.26) is a competition-diffusion system with non-local competition terms according to [19].

As it is pointed out in [19] one of the prominent characteristics of competition-diffusion systems for two species is the so-called comparison principle, which stems from the maximum principle. Owing to this property, the general theory of strongly order-preserving local semiflows, [12] implements to system (7.23)–(7.26), thereby providing a number of results on the dynamical structure of this system. To this end, we first introduce the comparison principle for system (7.23)–(7.26) as follows.

**Comparison Principle**: Let \((z, w), (\bar{z}, \bar{w})\) be solutions to (7.23)–(7.26) with initial data \((z_0, w_0), (\bar{z}_0, \bar{w}_0)\) respectively. Suppose \(\bar{z}_0(x) \geq z_0(x), \quad \bar{w}_0(x) \leq w_0(x)\) for all \(x \in \Omega\) and \(\bar{z}(x, t) \geq z(x, t), \quad \bar{w}(x, t) \leq w(x, t)\) for all \((x, t) \in \partial \Omega \times (0, T)\). Then \(\bar{w}(x, t) \geq w(x, t)\), and \(\bar{z}(x, t) \leq z(x, t)\) for all \((x, t) \in \Omega \times (0, T)\).

Motivated by the above comparison principle we determine, see also [19], the following order relation in the space \(C(\Omega) \times C(\Omega)\)

\[ \left( \frac{\bar{z}}{\bar{w}} \right) \succeq \left( \frac{z}{w} \right) \iff \bar{z}(x) \geq z(x) \quad \text{and} \quad \bar{w}(x) \leq w(x) \quad \text{for any} \quad x \in \Omega. \quad (7.51) \]

The strict form of relation (7.51) is defined as follows:

\[ \left( \frac{\bar{z}}{\bar{w}} \right) \succ \left( \frac{z}{w} \right) \iff \bar{z}(x) \geq z(x) \quad \text{and} \quad \bar{w}(x) \leq w(x) \quad \text{with} \quad \bar{z}(x) \gtrless z(x) \quad \text{and} \quad \bar{w}(x) \lhd w(x), \]

for any \(x \in \Omega\).

Owing to (7.50) the local semiflow designated by system (7.23)–(7.26) preserves the order relation defined above. Moreover, since we actually have

\[ \frac{\partial f(z, w)}{\partial w} < 0 \quad \text{and} \quad \frac{\partial g(z, w)}{\partial z} < 0, \]

then there finally holds that the local semiflow is strongly order-preserving with respect to the order relation provided above, [19].
Property (7.50) suggests that some interesting properties of the determined local semiflow hold as described by Theorem 7.3.2, see also [12, 19]. Before stating the result of this theorem we need to introduce the following definition.

**Definition 7.3.1** We say that

\[ \phi(x) = \begin{pmatrix} z(x) \\ w(x) \end{pmatrix}, \]

is a (time-independent) super-solution to (7.23)–(7.26) if the following inequalities are satisfied:

\[
-\Delta z \geq \frac{\lambda e^z}{\int_\Omega e^z \, dx}, \quad -\Delta w \leq \frac{\lambda e^w}{\int_\Omega e^w \, dx} \quad \text{in} \quad \Omega,
\]

\[ z \geq 0 \geq w \quad \text{on} \quad \partial \Omega. \]

In case the reverse inequalities are fulfilled then \( \phi \) is called a (time-independent) sub-solution. Once, the above inequalities are strict then we end up with strict super and sub-solutions.

**Theorem 7.3.2** The following statements hold true:

(i) The \( \omega \)-limit sets of almost bounded orbits of system (7.23)–(7.26) are contained in the set of the stationary solutions designated as:

\[
-\Delta v_1 = \frac{\lambda e^{v_1}}{\int_\Omega e^{v_1} \, dx}, \quad -\Delta v_2 = \frac{\lambda e^{v_2}}{\int_\Omega e^{v_2} \, dx} \quad \text{in} \quad \Omega, \quad (7.52)
\]

\[ v_1 = v_2 = 0 \quad \text{on} \quad \partial \Omega. \quad (7.53) \]

(ii) Any periodic orbit of system (7.23)–(7.26) is unstable.

(iii) Any unstable stationary solution

\[ v^*(x) = \begin{pmatrix} v_1^*(x) \\ v_2^*(x) \end{pmatrix}, \]

(which solves (7.52)–(7.53)) has a non-trivial unstable set, i.e. there exists

\[ \phi(x, t) = \begin{pmatrix} z(x, t) \\ w(x, t) \end{pmatrix} \neq v^*(x) = \begin{pmatrix} v_1^*(x) \\ v_2^*(x) \end{pmatrix}, \]

such that \( \phi(t) \to v^* \) as \( t \to -\infty \) in \( C(\Omega) \).

(iv) Assume that \( \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in C(\Omega) \times C(\Omega) \) be a strict super-solution and \( \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in C(\Omega) \times C(\Omega) \) be a strict sub-solution of (7.23)–(7.26) such that
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Let \( \xi(x) \gg \zeta(x) \) in \( \Omega \), then there exists a stable stationary solution \( v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \)

such that \( \xi(x) \leq v^*(x) \leq \zeta(x) \) in \( \Omega \), i.e.

\[
\xi_1(x) \leq v_1^*(x) \leq \zeta_1(x) \quad \text{and} \quad \xi_2(x) \leq v_2^*(x) \leq \zeta_2(x)
\]

for any \( x \) in \( \Omega \).

An immediate consequence of Theorem 7.3.2 is the following

**Corollary 7.3.3** If for system (7.23)–(7.26) we consider initial data \( \begin{pmatrix} z_0 \\ w_0 \end{pmatrix} = v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \),

where \( v^* \) is the stable stationary point provided by Theorem 7.3.2 (iv) then system (7.23)–(7.26) has a global-in-time solution.

**Proof** Indeed a straightforward application of the comparison principles entails that

\[
\xi(x) = \begin{pmatrix} \xi_1(x) \\ \xi_2(x) \end{pmatrix} \leq \begin{pmatrix} z(x,t) \\ v(x,t) \end{pmatrix} \leq \begin{pmatrix} \zeta_1(x) \\ \zeta_2(x) \end{pmatrix} = \zeta(x) \quad \text{for any} \quad x \in \Omega \quad \text{and for any} \quad t > 0.
\]

which evidently entails the existence of a global-in-time solution for problem (7.23)–(7.26).

**Remark 7.3.4** In case where

\[
\xi(x) = \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} \quad \text{and} \quad \zeta(x) = \begin{pmatrix} 0 \\ \psi(x) \end{pmatrix},
\]

for \( \psi \) being a steady-state solution of (7.15)–(7.17) then Corollary 7.3.3 guarantees the existence of a global-in-time solutions of this problem.

We close this section with a brief investigation of the steady-state problem (7.52)–(7.53) in the radial symmetric case, i.e. when \( \Omega = B_1(0) = \{ x \in \mathbb{R}^N \mid |x| < 1 \} \), for \( N \geq 3 \). We first observe that in this case the solutions of (7.52)–(7.53) are radially symmetric and thus problem (7.52)–(7.53) reduces to

\[
-\Delta_r v_1 = \frac{\lambda e^{v_1}}{N \omega_N \int_0^1 r^{N-1} e^{v_2(r)} \, dr}, \quad -\Delta_r v_2 = \frac{\lambda e^{v_2}}{N \omega_N \int_0^1 r^{N-1} e^{v_1(r)} \, dr} \quad \text{in} \quad (0, 1),
\]

\[
\left. \frac{\partial v_1}{\partial r} \right|_{r=0} = \left. \frac{\partial v_2}{\partial r} \right|_{r=0} = 0, \quad v_1(1) = v_2(1) = 0,
\]

(7.54)

where \( \Delta_r := r^{N-1} \frac{\partial^2}{\partial r^2} + (N-1) r^{N-2} \frac{\partial}{\partial r} \) and \( r = |x| \), recalling that \( \omega_N \) denotes the volume of the \( N \)-dimensional unit sphere. Let us set
then the non-local system (7.52)–(7.53) is reduced to the following local one

\[-\Delta_r v_1 = \sigma_1 e^{v_1}, \quad -\Delta_r v_2 = \sigma_2 e^{v_2} \text{ in } (0, 1),\]

(7.57)

\[\left. \frac{\partial v_1}{\partial r} \right|_{r=0} = \left. \frac{\partial v_2}{\partial r} \right|_{r=0} = 0, \quad v_1(1) = v_2(1) = 0, \quad v_1(1) = v_2(1) = 0,\]

(7.58)

and those two problems are equivalent through (7.56). Moreover, (7.56) also yields

\[\sigma_1 N \omega_N \int_0^1 r^{N-1} e^{v_1(r)} \, dr = \sigma_1 \lambda \quad \text{and} \quad \sigma_2 N \omega_N \int_0^1 r^{N-1} e^{v_2(r)} \, dr = \sigma_2 \lambda.\]

(7.59)

Remarkably, the solution set

\[\mathcal{C}_i = \{(\sigma_i, v_i) \mid v_i(x) \text{ is a classical solution to (7.57)–(7.58) for } \sigma_i > 0, \quad i = 1, 2,\]

is a one-dimensional open manifold with end points \((0, 0)\) and \((2(N - 2), \log \frac{1}{r})\), where the latter one is a weak solution of (7.57)–(7.58), see also [15, 20]. Therefore, the solution set

\[\mathcal{J}_r = \left\{(\lambda, v_1, v_2) \left| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ is a classical solution to (7.54)–(7.55) for } \lambda > 0 \right. \right\},\]

is a two-dimensional manifold with end points \((0, 0, 0)\) and \((2\sigma_N, \log \frac{1}{r}, \log \frac{1}{r})\), where \(\sigma_N = N \omega_N\) denotes the \((N - 1)\)-dimensional volume of the surface of the unit ball in \(\mathbb{R}^N\). Again the latter end point is a weak solution of (7.54)–(7.55).

Using Emden’s transformation

\[v_i(r) = w_i(\tau) - 2\tau + A, \quad r = Be^\tau, \quad B = \left(\frac{2(N - 2)}{\sigma_i e^A}\right)^{1/2}, \quad i = 1, 2,\]

(7.60)

then system (7.57)–(7.58) is transformed to

\[\dot{w}_i + (N - 2)\ddot{w}_i + 2(N - 2)(e^{w_i} - 1) = 0, \quad \lim_{\tau \to -\infty} w_i(\tau) = -\infty, \quad \lim_{\tau \to -\infty} \dot{w}_i(\tau) = 2, \quad i = 1, 2,\]

(7.61)

where \(\dot{w}_i = \frac{dw_i}{d\tau}, \quad \ddot{w}_i = \frac{d^2w_i}{d\tau^2}\), and by this transformation any \((\sigma_i, v_i) \in \mathcal{C}_i, \quad i = 1, 2,\) corresponds to a point \(P(w_i, \dot{w}_i) \in \mathcal{O} = \{(w(t), \dot{w}(t)) \mid t \in \mathbb{R}\} \).

Additionally, by virtue of (7.56), (7.61) we also have
\[ \dot{v}_i = \frac{\lambda}{\alpha_N} \sigma_i e^{\nu(r)} \] 
\[ \frac{\partial v}{\partial r} \bigg|_{r=1} = \alpha_N (2 - \dot{w}_i), \quad i = 1, 2, \quad (7.62) \]

which in conjunction with (7.59) entails
\[ 2 - \dot{w}_1 = \frac{\lambda}{\alpha_N} \sigma_1 = \frac{\lambda}{\alpha_N} e^{w_1 - w_2} \quad \text{and} \quad 2 - \dot{w}_2 = \frac{\lambda}{\alpha_N} \sigma_2 = \frac{\lambda}{\alpha_N} e^{w_2 - w_1}. \quad (7.63) \]

Now for any given \( 0 < \lambda < \lambda^* \) where
\[ \lambda^* := \sup\{\lambda > 0 : \text{problem (7.54) – (7.55) has a solution corresponding to } \lambda\}, \]
we consider a point \( P(w_0, \dot{w}_0) \in \mathcal{O} \) determined by \( 2 - \dot{w}_0 = \frac{\lambda}{\alpha_N} \). Assuming that \( w_1 > w_2 \), then there exists \( \gamma > 1 \) such that \( w_2 = w_1 - \log \gamma \), and thus
\[ 2 - \dot{w}_1 = \gamma \frac{\lambda}{\alpha_N}, \quad \text{and} \quad 2 - \dot{w}_2 = \frac{1}{\gamma} \frac{\lambda}{\alpha_N}, \quad (7.64) \]

by virtue of (7.62) and (7.63).

Writing \( w_2 = w_2(\gamma) \), we then have
\[ 2 - \dot{w}_2(\gamma) = \frac{1}{\gamma} \frac{\lambda}{\alpha_N}, \]
by (7.64), and if \( N \geq 10 \) the mapping \( \gamma \in (1, 2) \mapsto \dot{w}_2(\gamma) \) is monotone increasing and there holds
\[ \lim_{\gamma \uparrow 1} \dot{w}_2(\gamma) = \frac{\lambda}{\alpha_N} \quad \text{and} \quad \lim_{\gamma \downarrow 2} \dot{w}_2(\gamma) = 2. \]

Consequently, the following statement holds true

**Theorem 7.3.5** For any \( 0 < \lambda \leq \alpha_N \) and \( N \geq 10 \) there exists a solution \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) to (7.54)–(7.55) such that
\[ v_2 \leq v_0 \leq v_1, \quad (7.65) \]
where \( v_0 \) is a solution to the following
\[ -\Delta v = \frac{\lambda e^v}{N \omega_N \int_0^1 r^{N-1} e^{\nu(r)} \, dr}, \quad \text{in} \quad (0, 1), \]
\[ \frac{\partial v}{\partial r} \bigg|_{r=0} = 0, \quad v(1) = 0. \]

In case \( \lambda = \alpha_N \), then \( v_1 = v_2 = v_0 \), otherwise the inequalities in (7.65) are strict.
7.4 Miscellanea

The current section is devoted to several remarks. First of all we note that the Green’s function of the operator $-\Delta_D$ for $\Omega = B_1(0)$ is given explicitly by

$$G(x, x') = -\frac{1}{4\pi} \log \frac{|z - z'|^2}{|1 - \zeta|^2},$$

where $z$ and $z'$ are complex numbers corresponding to $x$ and $x'$, respectively. The latter implies

$$x \cdot \nabla_x G(x, x') + x' \cdot \nabla_{x'} G(x, x') = -\frac{1}{2\pi} \cdot \frac{1 - |\zeta|^2}{|1 - \zeta|^2},$$

for $\zeta = zz'$. We can then see that this function does not belong to $L^\infty(\Omega \times \Omega)$ in contrast with the corresponding function derived from the Green’s function to $-\Delta_{Ls}$, see also [32]. There is a similar difficulty in (7.5)–(7.8), and the control of the boundary blow-up points has not been completed even in the case of $(\varepsilon, \tau) = (1, 0).

Next, if $0 < \lambda < 8\pi$, the stationary problem (7.103) admits a unique solution [31]. Since (7.21) holds in this range of $\lambda$, then solution $v = v(x, t)$ to (7.15)–(7.17) converges uniformly to this steady solution. Now, what is anticipated for the case $\lambda = 8\pi$, $\Omega = B_1(0)$ is that an infinite-time blow-up occurs for the solution $v$, i.e., $T = +\infty$ and $v(x, t) \to 4 \log \frac{1}{|x|}$ locally uniformly in $x \in \Omega \setminus \{0\}$ as $t \uparrow +\infty$.

In the limiting case $\varepsilon = 0$ system (7.9)–(7.13) is described by

$$v_t = \Delta v + \lambda \left( \frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \quad \text{in} \quad \Omega \times (0, T),$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0, T),$$

with the smooth initial value $v_0 = v_0(x)$ satisfying $\int_\Omega v_0 dx = 0$.

In the case where either $0 < \lambda < 4\pi$ or $0 < \lambda < 8\pi$, $v_0 = v_0(|x|)$, and $\Omega = B_1(0)$, we obtain (7.21) similarly to the full system, see [2, 7, 23]. In contrast to the case of problem (7.30) the stationary problem of (7.66)–(7.67) has the trivial solution $v = 0$ for any $\lambda$. Furthermore, multiple existence of the stationary (non-radially symmetric) solution arises even for $0 < \lambda < 4\pi$ and $\Omega = B_1(0)$ [26, 32]. On the other hand, either $T = +\infty$ or (7.33) with $T < +\infty$ holds for problem (7.66)–(7.67), similarly to Theorem 7.2. There is, furthermore, a bifurcation of non-constant radially symmetric stationary solutions at $\lambda = \lambda^* > 8\pi$. Then, we conjecture that any $\lambda > 8\pi$ admits a radially symmetric stationary solution (possibly the trivial one $v = 0$), stable in the space of radially symmetric functions. The possibility of the occurrence of a mass concentration at the origin $x = 0$ as in (7.33) holding with $T < +\infty$ is left open for problem (7.66)–(7.67) even for $\lambda > 8\pi$ and $\Omega = B_1(0)$ in contrast to Theorem 7.2.
A simple blow-up criterion is obtained for the semilinear parabolic equation

\[ v_t = \Delta v + |v|^{q-1} v \quad \text{in} \quad \Omega \times (0, T), \quad v = 0 \quad \text{on} \quad \partial \Omega \times (0, T), \]

with \( 1 < q < \infty \). In fact, this equation admits the properties

\[
\frac{d}{dt} J(v(\cdot, t)) \leq 0, \\
\frac{1}{4} \frac{d}{dt} \|v(\cdot, t)\|_2^2 = -J(v(\cdot, t)) + \left( \frac{1}{2} - \frac{1}{q + 1} \right) \|v(\cdot, t)\|_{q+1}^{q+1},
\]

for

\[ J(v) = \frac{1}{2} \|\nabla v\|_2^2 - \frac{1}{q + 1} \|v\|_{q+1}^{q+1}. \]

Using the above, we can infer \( T < +\infty \) by \( J(v_0) \leq 0 \). The same argument is valid to the following non-local problem

\[ v_t = \Delta v + \frac{\lambda e^v}{(\int_{\Omega} e^v)^p} \quad \text{in} \quad \Omega \times (0, T), \]

\[ v(x, t) = 0 \quad \text{on} \quad \partial \Omega \times (0, T), \]

\[ v(x, 0) = v_0(x) \quad \text{in} \quad \Omega, \]

with \( 0 < p < 1 \), see [1], but this is not the case for problem (7.15)–(7.17).

In fact, problem (7.15)–(7.17) can be written in the form (7.35) with \( u = \frac{\lambda e^v}{(\int_{\Omega} e^v)^p} \) satisfying \( \|u\|_1 = \lambda \). Set \( \mathcal{J}_\lambda(v) = L(u, v) \), where

\[ \mathcal{J}_\lambda(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \left( \int_{\Omega} e^v \right) + \lambda (\log \lambda - 1), \]

\[ L(u, v) = \int_{\Omega} u (\log u - 1) + \frac{1}{2} \|\nabla v\|_2^2 - \langle v, u \rangle. \]

Then, it holds that

\[
\frac{1}{4} \frac{d}{dt} \|v\|_2^2 = -L(u, v) + \int_{\Omega} u (\log u - 1) - \frac{1}{2} uv \, dx \\
= -\mathcal{J}_\lambda(v) + K(u, v),
\]

with

\[ K(u, v) = \int_{\Omega} u (\log u - 1) - \frac{1}{2} uv \, dx \]

\[ \geq K|_{u = \lambda e^{v/2}} = -\lambda \log \int_{\Omega} e^{v/2} + \lambda (\log \lambda - 1), \]
\[
\frac{1}{4} \frac{d}{dt} \|v(\cdot, t)\|_2^2 \geq -\mathcal{J}_\lambda(v(\cdot, t)) - \lambda \log \left( \int_\Omega e^{v(\cdot, t)/2} \right) + \lambda (\log \lambda - 1).
\]

In spite of
\[
\frac{d}{dt} \mathcal{J}_\lambda(v(\cdot, t)) \leq 0,
\]
the above inequality is not sufficient to guarantee \(T < +\infty\) because of the negativity of the second term on its right-hand side.

\section*{References}