Chapter 5
Inheritance in Multivariate Subordination

We now study inheritance of $L_m$ property or strict stability from subordinator to subordinated in multivariate subordination. In order to observe this inheritance, we have to assume strict stability of the distribution at each $s \in K$ of a $K$-parameter subordinand $\{X_s : s \in K\}$. Section 5.1 gives results and examples. Section 5.2 discusses some generalization where the defining condition of selfdecomposability or stability for distributions on $\mathbb{R}^d$ involves a $d \times d$ matrix $Q$. This is called operator generalization.

5.1 Inheritance of $L_m$ Property and Strict Stability

We begin with the following theorem and examples in the usual subordination.

**Theorem 5.1** Suppose that $\{X_t : t \geq 0\}$ is a strictly $\alpha$-stable process on $\mathbb{R}^d$, $\{Z_t : t \geq 0\}$ is a subordinator, and they are independent. Let $\{Y_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ constructed from $\{X_t\}$ by subordination by $\{Z_t\}$.

(i) If $\{Z_t\}$ is selfdecomposable, then $\{Y_t\}$ is selfdecomposable.
(ii) More generally, let $m \in \{0, 1, \ldots, \infty\}$. If $\{Z_t\}$ is of class $L_m(\mathbb{R})$, then $\{Y_t\}$ is of class $L_m(\mathbb{R}^d)$.
(iii) If $\{Z_t\}$ is strictly $\beta$-stable, then $\{Y_t\}$ is strictly $\alpha\beta$-stable.

Halgreen [30] (1979) and Ismail and Kelker [36] (1979) proved part of these results. Proof of Theorem 5.1 will be given as a special case of Theorem 5.9.

**Example 5.2** Let $0 < \alpha < 1$. Let $\{Y_t\}$ be the Lévy process on $\mathbb{R}$ subordinate to a strictly $\alpha$-stable increasing process $\{X_t\}$ on $\mathbb{R}$ with $Ee^{-uX_t} = e^{-tu^\alpha}$, $u \geq 0$, by a $\Gamma$-process $\{Z_t\}$ with $EZ_1 = 1$. Then

$$P[Y_1 \leq x] = 1 - E_\alpha(-x^\alpha), \quad x \geq 0,$$

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where $E_\alpha(x)$ is the Mittag–Leffler function $E_\alpha(x) = \sum_{n=0}^{\infty} x^n / \Gamma(n\alpha + 1)$, and

$$P[Y_t \leq x] = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(t+n)}{n! \Gamma(t) \Gamma(1+\alpha(t+n))} x^{\alpha(t+n)}, \quad x \geq 0.$$ 

By Theorem 5.1, $\mathcal{L}(Y_t)$ is selfdecomposable. See Pillai [79] (1990) or Sato [93, E 34.4] (1999).

Example 5.3 Let $0 < \alpha \leq 2$. Let $\{Y_t\}$ be the Lévy process subordinate to a symmetric $\alpha$-stable process $\{X_t\}$ on $\mathbb{R}$ with $E e^{izX_t} = e^{-t|z|^{\alpha}}$ by a $\Gamma$-process $\{Z_t\}$ with $EZ_1 = 1/q$, $q > 0$. Then

$$E e^{izY_t} = (1 + q^{-1}|z|^{\alpha})^{-t}, \quad z \in \mathbb{R},$$

where $\mathcal{L}(Y_1)$ is Linnik distribution or geometric stable distribution (Example 4.7). Theorem 5.1 shows that $\mathcal{L}(Y_t)$ is selfdecomposable.

In the definitions and examples below, we use $\gamma, \delta, \lambda, \chi, \psi$ for parameters of some special distributions, keeping a customary usage.

Definition 5.4 The distribution

$$\mu_{\gamma,\delta}(dx) = (2\pi)^{-1/2} \delta e^{\gamma \delta} x^{-3/2} e^{-(\delta^2 x^{-1} + \gamma^2 x)/2} 1_{(0,\infty)}(x) dx$$

with parameters $\gamma > 0, \delta > 0$ is called inverse Gaussian distribution.

The Laplace transform $L_{\mu_{\gamma,\delta}}(u), u \geq 0$, of $\mu_{\gamma,\delta}$ is

$$L_{\mu_{\gamma,\delta}}(u) = \int_{(0,\infty)} e^{-ux} \mu_{\gamma,\delta}(dx) = \exp \left[ -\delta \left( \sqrt{2u + \gamma^2} - \gamma \right) \right]$$

$$= \exp \left[ 2^{-1} \pi^{-1/2} \delta \int_{0}^{\infty} (e^{-(2u+\gamma^2)x} - 1) x^{-3/2} dx + \gamma \delta \right]$$

$$= \exp \left[ 2^{-1} \pi^{-1/2} \delta \int_{0}^{\infty} (e^{-2ux} - 1) x^{-3/2} e^{-\gamma^2 x} dx \right]$$

$$= \exp \left[ (2\pi)^{-1/2} \delta \int_{0}^{\infty} (e^{-ux} - 1) x^{-3/2} e^{-\gamma^2 x/2} dx \right].$$

The last formula shows that $\mu_{\gamma,\delta}$ is infinitely divisible with Lévy measure density

$$(2\pi)^{-1/2} \delta x^{-3/2} e^{-\gamma^2 x/2}$$

on $(0, \infty)$. Hence $\mu_{\gamma,\delta}$ is selfdecomposable by Theorem 1.34.

For every $\lambda \in \mathbb{R}$ we denote by $K_\lambda$ the modified Bessel function of order $\lambda$ given by (4.9), (4.10) of [93, p. 21].
Example 5.5 Let \( \{ Z_t \} \) be a subordinator with \( \mathcal{L}(Z_1) = \mu_{\gamma, \delta} \). Then \( \mathcal{L}(Z_t) = \mu_{\gamma, t\delta} \). Let \( \{ Y_t \} \) be the Lévy process subordinate to Brownian motion \( \{ X_t \} \) on \( \mathbb{R} \) by \( \{ Z_t \} \). Then

\[
\mathbb{P}[Y_t \in B] = \int_0^\infty \mu_{\gamma, t\delta}(ds) \int_B (2\pi s)^{-1/2} e^{-x^2/(2s)} dx
\]

\[
= (2\pi)^{-1} \delta e^{t\gamma \delta} \int_B dx \int_0^\infty s^{-2} e^{-(x^2+t^2\delta^2)/(2s)-(\gamma^2 s/2)} ds
\]

\[
= (4\pi)^{-1} \delta e^{t\gamma \delta} \int_B dx \int_0^\infty u^{-2} e^{-(\gamma^2 (x^2+t^2\delta^2)/(4u)) - u} du
\]

\[
= \int_B \frac{\gamma e^{t\gamma \delta}}{\pi \sqrt{1 + (x/(t\gamma))^2}} K_1 \left( t\gamma \delta \sqrt{1 + (x/(t\gamma))^2} \right) dx,
\]

where \( K_1 \) is the modified Bessel function of order 1. This shows that \( \mathcal{L}(Y_t) \) is a special case of the \emph{normal inverse Gaussian distribution} defined by Barndorff-Nielsen [5] (1997). By Theorem 5.1, it is selfdecomposable. By Theorem 4.3 its characteristic function is

\[
E e^{iz Y_t} = e^{t\Psi(-z^2/2)} = \exp \left[ -t\delta \left( \sqrt{z^2 + \gamma^2} - \gamma \right) \right]
\]

with \( \Psi(w) = -\delta \left( \sqrt{-2w + \gamma^2} - \gamma \right) \).

Definition 5.6 The distribution

\[
\mu_{\lambda, \chi, \psi}(dx) = c x^{\lambda-1} e^{-(\chi x^{-1} + \psi x)/2} 1_{(0,\infty)}(x) dx
\]

is called \emph{generalized inverse Gaussian distribution} with parameters \( \lambda, \chi, \psi \). Here \( c \) is a normalizing constant. The domain of the parameters is given by \( \{ \lambda < 0, \chi > 0, \psi \geq 0 \}, \{ \lambda = 0, \chi > 0, \psi > 0 \}, \) and \( \{ \lambda > 0, \chi \geq 0, \psi > 0 \} \).

The Laplace transform \( L_{\mu_{\lambda, \chi, \psi}}(u), u \geq 0, \) of \( \mu_{\lambda, \chi, \psi} \) is

\[
L_{\mu_{\lambda, \chi, \psi}}(u) = \begin{cases} \left( \frac{\psi}{\psi + 2u} \right)^{\lambda/2} K_\lambda \left( \sqrt{\chi (\psi + 2u)} \right) K_\lambda \left( \sqrt{\chi \psi} \right) & \text{if } \lambda > 0 \text{ and } \psi > 0 \\ \frac{2^{1+\lambda/2} K_\lambda \left( \sqrt{2\chi u} \right)}{\Gamma(-\lambda)(\chi u)^{\lambda/2}} & \text{if } \lambda < 0, \chi > 0, \text{ and } \psi = 0, \end{cases}
\]

where \( K_\lambda \) is the modified Bessel function of order \( \lambda \). It is known that \( \mu_{\lambda, \chi, \psi} \) is infinitely divisible and, moreover, selfdecomposable [93, E. 34.13]. It belongs to the smaller class \( GGC \) called \emph{generalized \( \Gamma \)-convolutions}, which means that it is
the limit of a sequence of convolutions of $\Gamma$-distributions. See Halgreen [30] (1979). Concerning this class, see Notes at end of Chap. 2.

In order to extend Theorem 5.1 to multivariate subordination, we prepare two lemmas.

**Lemma 5.7** Let $K$ be a cone in $\mathbb{R}^N$ and $\{X_s : s \in K\}$ a $K$-parameter Lévy process on $\mathbb{R}^d$. Let $0 < \alpha \leq 2$. Then $\mathcal{L}(X_s) \in \mathcal{G}_\alpha^0$ if and only if $X_{ts} \overset{d}{=} t^{1/\alpha} X_s$ for every $t > 0$.

**Proof** Let $\mu_s = \mathcal{L}(X_s)$. The meaning of $\mu_s \in \mathcal{G}_\alpha^0$ is that $\mu_s \in ID$ and $\hat{\mu}_s(z) = \hat{\mu}_s(t^{1/\alpha} z)$ for $t > 0$. See Definition 1.23 and Proposition 1.22. Since, by Lemma 4.21, $\{X_s : t \geq 0\}$ is a Lévy process, $\hat{\mu}_{ts}(z) = \hat{\mu}_s(z)^t$. Hence the condition is written as $X_{ts} \overset{d}{=} t^{1/\alpha} X_s$. \hfill $\blacksquare$

**Lemma 5.8** Let $\{Z_t\}$ be a $K$-valued subordinator such that $\mathcal{L}(Z_t) \in L_0(\mathbb{R}^N)$ for $t \geq 0$. Let $\Psi(w)$ be the function in (4.11). For $b > 1$ define $\Psi_b(w)$ as

$$
\Psi(w) = \Psi(b^{-1}w) + \Psi_b(w).
$$

(5.1)

Then $e^{i\Psi_b(iz)}$, $z \in \mathbb{R}^N$, is the characteristic function of a $K$-valued subordinator $\{Z_t^{(b)}\}$. Let $m \geq 1$. Then $\mathcal{L}(Z_t) \in L_m$ for $t \geq 0$ if and only if $\mathcal{L}(Z_t^{(b)}) \in L_{m-1}$ for $t \geq 0$ and $b > 1$.

**Proof** Let $\mu = \mathcal{L}(Z_1)$ with generating triplet $(A, \nu, \gamma)$. Its characteristic function is

$$
\hat{\mu}(z) = e^{i\Psi(iz)}, z \in \mathbb{R}^N.
$$

If $b > 1$, then by selfdecomposability there is a distribution $\rho_b$ such that

$$
\hat{\mu}(z) = \hat{\mu}(b^{-1}z) \hat{\rho}_b(z).
$$

Let $\mu_b$ be such that $\hat{\rho}_b(z) = \hat{\mu}(b^{-1}z)$. Then $\mu = \mu_b \ast \rho_b$ and, by Proposition 1.13, $\mu_b$ and $\rho_b$ are in $ID$. Let $(\tilde{A}_b, \tilde{\nu}_b, \tilde{\gamma}_b)$ and $(A_b, \nu_b, \gamma_b)$ be the generating triplets of $\mu_b$ and $\rho_b$, respectively. Then $A = \tilde{A}_b + A_b$, $\nu = \tilde{\nu}_b + \nu_b$, and $\gamma = \tilde{\gamma}_b + \gamma_b$. Hence $\nu_b \leq \nu$. By Theorem 4.11, $A = 0$, $\nu(\mathbb{R}^N \setminus K) = 0$, $\int_{|s| \leq 1} |s| \nu(ds) < \infty$, and $\gamma^0 \in K$. Therefore $\nu_b(\mathbb{R}^N \setminus K) = 0$, and $\int_{|s| \leq 1} |s| \nu_b(ds) < \infty$. Also $A_b = 0$, as $0 \leq \langle z, A_b z \rangle \leq \langle z, A z \rangle = 0$. Further, their drifts are related as $\gamma^0 = \tilde{\gamma}_b^0 + \gamma_b^0$ and $\tilde{\gamma}_b^0 = b^{-1} \gamma^0$. Thus $\gamma_b^0 = (1 - b^{-1})^{-1} \gamma^0 \in K$. Then, by Theorem 4.11, a Lévy process $\{Z_t^{(b)}\}$ with $\mathcal{L}(Z_t^{(b)}) = \rho_b$ is a $K$-valued subordinator. Its characteristic function equals $(\hat{\rho}_b(z)^{-1})^{-1} = e^{i\Psi_b(iz)}$. Finally, $\mathcal{L}(Z_t)$ is of class $L_m$ if and only if, for each $b > 1$, $\rho_b \in L_{m-1}$, that is, $\mathcal{L}(Z_t^{(b)})$ is of class $L_{m-1}$. \hfill $\blacksquare$

**Theorem 5.9** Let $K$ be a cone in $\mathbb{R}^N$ and $0 < \alpha \leq 2$. Let $\{Z_t : t \geq 0\}$ be a $K$-valued subordinator and $\{X_s : s \in K\}$ a $K$-parameter Lévy process on $\mathbb{R}^d$ such that $\mathcal{L}(X_s) \in \mathcal{G}_\alpha^0$ for all $s \in K$. Assume that they are independent. Let $\{Y_t : t \geq 0\}$ be the Lévy process on $\mathbb{R}^d$ constructed from $\{X_t\}$ and $\{Z_t\}$ by multivariate subordination of Definition 4.24.
5.1 Inheritance of $L_m$ Property and Strict Stability

(i) If $\{Z_t\}$ is selfdecomposable, then $\{Y_t\}$ is selfdecomposable.

(ii) Let $m \in \{0, 1, \ldots, \infty\}$. If $\{Z_t\}$ is of class $L_m(\mathbb{R}^d)$, then $\{Y_t\}$ is of class $L_m(\mathbb{R}^d)$.

(iii) Let $0 < \beta \leq 1$. If $\mathcal{L}(Z_t) \in \mathcal{S}_\beta^0$ for all $t \geq 0$, then $\mathcal{L}(Y_t) \in \mathcal{S}_{\alpha\beta}^0$ for all $t \geq 0$.

Proof Let $\mu_s = \mathcal{L}(X_s)$.

(i) Let $\{Z_t\}$ be selfdecomposable. Then $\mathcal{L}(Z_t) \in L_0$ for all $t \geq 0$. Using Lemma 5.8 and its notation, we have

$$Z_t \overset{d}{=} b^{-1}Z_t + Z_t^{(b)},$$

where $b^{-1}Z_t$ and $Z_t^{(b)}$ are independent. Then,

$$E e^{i(z, Y_t)} = E e^{i(b^{-1}\alpha z, Y_t)} E \left[ \hat{\mu}_{Z_t^{(b)}}(z) \right]. \quad (5.2)$$

Indeed we have, using Lemma 4.21 (i) and Lemma 5.7,

$$E e^{i(z, Y_t)} = E \left[ \left( E e^{i(z, X_s)} \right)_{s=Z_t} \right] = E \left[ \hat{\mu}_{Z_t}(z) \right] = E \left[ \hat{\mu}_{b^{-1}Z_t + Z_t^{(b)}}(z) \right]$$

$$= E \left[ \hat{\mu}_{Z_t}(z) \right] = E \left[ \hat{\mu}_{b^{-1}Z_t}(z) E \left[ \hat{\mu}_{Z_t^{(b)}}(z) \right] \right]$$

which is the right-hand side of (5.2). Notice that $b^{1/\alpha}$ can be an arbitrary real bigger than 1 and $E \left[ \hat{\mu}_{Z_t^{(b)}}(z) \right]$ is the characteristic function of a subordinated process by Lemma 5.8. This shows that $\{Y_t\}$ is selfdecomposable.

(ii) By induction. If $m = 0$, then the assertion is true by (i). Suppose that the assertion is true for $m - 1$ in place of $m$. Let $\{Z_t\}$ be of class $L_m$, that is, $\mathcal{L}(Z_t) \in L_m$ for $t \geq 0$. Then $\{Z_t^{(b)}\}$ is a $K$-valued subordinator of class $L_m$ by Lemma 5.8. Hence $E \left[ \hat{\mu}_{Z_t^{(b)}}(z) \right]$ is a characteristic function of class $L_{m-1}$. Thus $\mathcal{L}(Y_t) \in L_m$.

(iii) Let $\mathcal{L}(Z_t) \in \mathcal{S}_\beta^0$ for $t \geq 0$. Then $Z_{at} \overset{d}{=} a^{1/\beta}Z_t$. Therefore, using Lemma 5.7,

$$E e^{i(z, Y_{at})} = E \left[ \left( E e^{i(z, X_s)} \right)_{s=Z_{at}} \right] = E \left[ \left( E e^{i(z, X_s)} \right)_{s=a^{1/\beta}Z_t} \right]$$

$$= E \left[ \hat{\mu}_{a^{1/\beta}Z_t}(z) \right] = E \left[ \hat{\mu}_{Z_t}(a^{1/(\alpha\beta)}z) \right] = E \left[ e^{i(z, a^{1/(\alpha\beta)}Y_t)} \right].$$

Thus $Y_{at} \overset{d}{=} a^{1/(\alpha\beta)}Y_t$ for any $a > 0$. ■
When $d = 1$, Theorem 5.1 can be generalized to the case where $\{X_t : t \geq 0\}$ is Brownian motion with non-zero drift on $\mathbb{R}$. This is 2-stable, but not strictly 2-stable. So the assumption in Theorem 5.1 is not satisfied. Nevertheless, selfdecomposability is inherited as follows.

**Theorem 5.10** Let $\{X_t : t \geq 0\}$ be Brownian motion with drift $\gamma$ on $\mathbb{R}$. That is,
\[
E e^{izX_t} = e^{t(-z^2/2+i\gamma z)}, \quad z \in \mathbb{R}.
\]

Let $\{Y_t\}$ be a Lévy process subordinate to $\{X_t\}$ by $\{Z_t\}$. If $\{Z_t\}$ is selfdecomposable, then $\{Y_t\}$ is selfdecomposable.


**Remark 5.11** There arises the question whether Theorem 5.10 can be extended to the case where $\{X_t\}$ is an $\alpha$-stable, not strictly $\alpha$-stable process with $0 < \alpha < 2$ on $\mathbb{R}$. Ramachandran’s paper [80] (1997) contains an answer to this question. Namely, if $1 < \alpha < 2$, then there are an $\alpha$-stable, not strictly $\alpha$-stable process $\{X_t\}$ on $\mathbb{R}$ and a selfdecomposable subordinator $\{Z_t\}$ such that the Lévy process $\{Y_t\}$ subordinate to $\{X_t\}$ by $\{Z_t\}$ is not selfdecomposable. Specifically, Ramachandran shows that if $E e^{izX_t} = e^{t(-c|z|^{\alpha}+i\gamma z)}$ with $1 < \alpha < 2$, $c > 0$, and $\gamma \neq 0$ and $\{Z_t\}$ is $\Gamma$-process with parameter $\lambda > 0$ (a special case of Example 4.7), then $\{Y_t\}$ is not selfdecomposable. The question in the case $0 < \alpha \leq 1$ is still open in the authors’ knowledge.

**Remark 5.12** If $d \geq 2$, then the situation is quite different and Theorem 5.10 cannot be generalized. It is known that, for $d \geq 2$, a Lévy process $\{Y_t\}$ on $\mathbb{R}^d$ subordinate to Brownian motion with drift, $\{X_t\}$, by a selfdecomposable subordinator $\{Z_t\}$ is not necessarily selfdecomposable. Even if $\mathcal{L}(Z_1)$ is a generalized $\Gamma$-convolution, $\{Y_t\}$ is not necessarily selfdecomposable.

**Definition 5.13** The distribution
\[
\mu(dx) = c \exp \left(-a\sqrt{1+x^2}+bx\right) dx
\]
on $\mathbb{R}$ with parameters $a$, $b$ satisfying $a > 0$ and $|b| < a$ or a scale change of this distribution is called *hyperbolic distribution*. Here $c$ is a normalizing constant.

The distribution
\[
\mu(dx) = c \left(\sqrt{1+x^2}\right)^{\lambda-(1/2)} K_{\lambda-(1/2)} \left(a\sqrt{1+x^2}\right) e^{bx}
\]
on $\mathbb{R}$ or its scale change, where $c$ is normalizing constant, is called *generalized hyperbolic distribution*. Here the domain of parameters is given by $\{\lambda \geq 0, a >$
0, \(|b| < a\) and \((\lambda < 0, a > 0, |b| \leq a\). This distribution reduces to the hyperbolic distribution if \(\lambda = 1\).

Example 5.14 Let \(\{X_t\}\) be Brownian motion with drift \(\gamma\) being zero or non-zero and let \(\{Z_t\}\) be the subordinator with \(\mathcal{L}(Z_1)\) being generalized inverse Gaussian \(\mu_{\lambda, \chi, \psi}\) with \(\lambda = 1, \chi > 0, \psi > 0\). Let us calculate the distribution at \(t = 1\) for the Lévy process \(\{Y_t\}\) subordinate to \(\{X_t\}\) by \(\{Z_t\}\):

\[
P [Y_1 \in B] = c \int_0^\infty e^{-(\chi s^{-1} + \psi s)/2} ds \int_B e^{-(x-s\gamma)/2s} dx
\]

by the calculation in Example 2.13 of [93]. Hence \(\mathcal{L}(Y_1)\) is a hyperbolic distribution with \(a = \sqrt{\chi(\psi + \gamma)}\) and \(b = \sqrt{\chi \gamma}\).

More generally if we assume that \(\mathcal{L}(Z_1)\) is generalized inverse Gaussian \(\mu_{\lambda, \chi, \psi}\), then \(\mathcal{L}(Y_1)\) is generalized hyperbolic distribution. For a proof, use the formula (30.28) of [93] for modified Bessel functions. It follows from Theorem 5.1 (if \(\gamma = 0\)) and Theorem 5.10 (if \(\gamma \neq 0\)) that generalized hyperbolic distributions are selfdecomposable.

5.2 Operator Generalization

For distributions on \(\mathbb{R}^d, d \geq 2\), the concepts of stability, selfdecomposability, and \(L_m\) property are generalized to the situation where multiplication by positive real numbers is replaced by multiplication by matrices of the form \(bQ\).

For a set \(J \subset \mathbb{R}\) let \(M_J(d)\) be the set of real \(d \times d\) matrices all of whose eigenvalues have real parts in \(J\). Let \(Q \in M_{(0, \infty)}(d)\).

Definition 5.15 A distribution \(\mu\) on \(\mathbb{R}^d\) is called \(Q\)-selfdecomposable if, for every \(b > 1\), there is \(\rho_b \in \mathcal{P}(\mathbb{R}^d)\) such that

\[
\widehat{\mu}(z) = \widehat{\mu}(b^{-Q^T}z)\widehat{\rho_b}(z), \quad z \in \mathbb{R}^d,
\]

where \(Q^T\) is the transpose of \(Q\) and \(b^{-Q^T}\) is a \(d \times d\) matrix defined by

\[
b^{-Q^T} = e^{-(\log b)Q^T} = \sum_{n=0}^{\infty} (n!)^{-1} (-\log b)^n (Q^T)^n.
\]

The class of all \(Q\)-selfdecomposable distributions on \(\mathbb{R}^d\) is denoted by \(L_0(Q)\). For \(m = 1, 2, \ldots\) the class \(L_m(Q)\) is defined to be the class of distributions \(\mu\) on \(\mathbb{R}^d\)
such that, for every $b > 1$, there exists $\rho_b \in L_{m-1}(Q)$ satisfying (5.3). Define $L_\infty(Q) = \bigcap_{m<\infty} L_m(Q)$.

It follows that $L_m(Q) = L_m(aQ)$ for any $a > 0$ and $m = 0, 1, \ldots, \infty$.

**Proposition 5.16** The classes just introduced form nested classes

$$\text{ID} \supset L_0(Q) \supset L_1(Q) \supset \cdots \supset L_\infty(Q).$$

(5.4)

Proof can be given analogously to the proofs of Propositions 1.13 and 1.15. See Jurek [39] (1983a) and Sato and Yamazato [111] (1985).

**Definition 5.17** A distribution $\mu$ on $\mathbb{R}^d$ is called $Q$-stable if, for every $n \in \mathbb{N}$, there is $c \in \mathbb{R}^d$ such that

$$\hat{\mu}(z)^n = \hat{\mu}(nQ^\top z)e^{i(c,z)}, \quad z \in \mathbb{R}^d.$$ (5.5)

It is called strictly $Q$-stable if, for all $n$,

$$\hat{\mu}(z)^n = \hat{\mu}(nQ^\top z), \quad z \in \mathbb{R}^d.$$ (5.6)

Let $\mathcal{S}_Q$ be the class of $Q$-stable distributions on $\mathbb{R}^d$. Let $\mathcal{S}^0_Q$ be the class of strictly $Q$-stable distributions on $\mathbb{R}^d$.

Here we are using the usual terminology, but it is not harmonious with the usage of the word $\alpha$-stable; $\mu$ is $\alpha$-stable if and only if it is $(\alpha^{-1}I)$-stable, where $I$ is the identity matrix. Similarly to the $\alpha$-stable case, we have the following.

**Proposition 5.18** A distribution $\mu$ is $Q$-stable if and only if $\mu \in \text{ID}$ and, for every $t > 0$, there is $c \in \mathbb{R}^d$ such that

$$\hat{\mu}(z)^t = \hat{\mu}(tQ^\top z)e^{i(c,z)}.$$ (5.7)

A distribution $\mu$ is strictly $Q$-stable if and only if $\mu \in \text{ID}$ and, for every $t > 0$,

$$\hat{\mu}(z)^t = \hat{\mu}(tQ^\top z).$$ (5.8)

Proof is like that of Proposition 1.21.

**Remark 5.19** If $\mu \in \mathcal{S}_Q$ for some $Q \in M((0,\infty))(d)$, then $\mu$ is called operator stable and sometimes $Q$ is called exponent of operator stability of $\mu$. But, in general, $Q$ is not uniquely determined by $\mu$; see Hudson and Mason [35] (1981) and Sato [86] (1985). If $\mu \in L_0(Q)$ for some $Q \in M((0,\infty))(d)$, then $\mu$ is called operator selfdecomposable.

**Remark 5.20** Operator stable and operator selfdecomposable distributions appear in a natural way when we study limit theorems for sums of a sequence of independent random vectors, allowing linear transformations (matrix multiplications) of partial sums. Basic papers are Sharpe [113] (1969) and Urbanik [128] (1972a).
Proposition 5.21 Suppose that \( \mu \) is \( Q \)-stable and nondegenerate on \( \mathbb{R}^d \). Then \( Q \) must be in \( M_{(1/2, \infty)}(d) \) and, moreover, any eigenvalue of \( Q \) with real part 1/2 is a simple root of the minimal polynomial of \( Q \); \( \mu \) is Gaussian if and only if \( Q \in M_{(1/2)}(d) \); \( \mu \) is purely non-Gaussian if and only if \( Q \in M_{(1/2, \infty)}(d) \).

This is by Sharpe [113] (1969).

Definition 5.22 For \( Q \in M_{(0, \infty)}(d) \), let \( \mathcal{S}(Q) \) denote the union of \( \mathcal{S}_{aQ} \) over all \( a > 0 \); let \( \mathcal{S}^0(Q) \) denote the union of \( \mathcal{S}_{aQ}^0 \) over all \( a > 0 \). The relation with \( \mathcal{S} \) and \( \mathcal{S}^0 \) in Definition 1.19 is that \( \mathcal{S} = \mathcal{S}(I) \) and \( \mathcal{S}^0 = \mathcal{S}^0(I) \).

The class \( \mathcal{S}(Q) \) is a subclass of \( L_\infty(Q) \). Moreover, we have the following.

Proposition 5.23 The class \( L_\infty(Q) \) is the smallest class containing \( \mathcal{S}(Q) \) and closed under convolution and weak convergence.


Definition 5.24 A Lévy process \( \{ X_t : t \geq 0 \} \) is called \( Q \)-selfdecomposable, \( Q \)-stable, or of class \( L_m(Q) \), respectively, if \( \mathcal{L}(X_t) \) (or, equivalently, \( \mathcal{L}(X_t) \) for every \( t \geq 0 \)) is \( Q \)-selfdecomposable, \( Q \)-stable, or of class \( L_m(Q) \).

Here are results on the inheritance of operator selfdecomposability, \( L_m(Q) \) property, and strict operator stability in some cases. These partially extend Theorem 5.9. Propositions 5.21 and 5.23 are not used in the proof.

Let \( N \) and \( d \) be positive integers satisfying \( d \geq N \geq 1 \). Let \( d_j, 1 \leq j \leq N \), be positive integers such that \( d_1 + \cdots + d_N = d \). Every \( x \in \mathbb{R}^d \) is expressed as \( x = (x_j)_{1 \leq j \leq N} \) with \( x_j \in \mathbb{R}^{d_j} \). We call \( x_j \) the \( j \)-th component-block of \( x \). The \( j \)-th component-block of \( X_t \) is denoted by \( (X_t)_j \). As in Sect. 4.3, we use the unit vectors \( e_k = (\delta_{kj})_{1 \leq j \leq N}, k = 1, \ldots, N, \) in \( \mathbb{R}^N \).

Theorem 5.25 Suppose that \( \{ X_s : s \in \mathbb{R}^N_+ \} \) is a given \( \mathbb{R}^N_+ \)-parameter Lévy process on \( \mathbb{R}^d \) with the following structure: for each \( j = 1, \ldots, N \),

\[
(X_{tej})_k = 0 \quad \text{for all } k \neq j. \tag{5.9}
\]

Suppose that \( \{ Z_t : t \geq 0 \} \) is a given \( \mathbb{R}^N_+ \)-valued subordinator and let \( \{ Y_t : t \geq 0 \} \) be a Lévy process on \( \mathbb{R}^d \) obtained by multivariate subordination from \( \{ X_s \} \) and \( \{ Z_t \} \). That is, \( \{ X_s \} \) and \( \{ Z_t \} \) are independent and \( Y_t = X_{Z_t} \). Let \( Q_j \in M_{(1/2, \infty)}(d_j) \) and \( c_j > 0 \) for \( 1 \leq j \leq N \), and let \( C = \text{diag}(c_1, \ldots, c_N) \). Assume that, for each \( j \), \( \mathcal{L}((X_{tej})_j) \) is strictly \( Q_j \)-stable. Let

\[
D = \text{diag}(c_1 Q_1, \ldots, c_N Q_N) \in M_{(0, \infty)}(d).
\]

(i) If \( \{ Z_t : t \geq 0 \} \) is \( C \)-selfdecomposable, then \( \{ Y_t : t \geq 0 \} \) is \( D \)-selfdecomposable.

(ii) More generally, let \( m \in \{ 0, 1, \ldots, \infty \} \). If \( \{ Z_t : t \geq 0 \} \) is of class \( L_m(C) \) on \( \mathbb{R}^N \), then \( \{ Y_t : t \geq 0 \} \) is of class \( L_m(D) \) on \( \mathbb{R}^d \).
(iii) If \( \{Z_t : t \geq 0\} \) is strictly \( C \)-stable, then \( \{Y_t : t \geq 0\} \) is strictly \( D \)-stable.

Here \( \text{diag}(c_1, \ldots, c_N) \) denotes the diagonal matrix with diagonal entries \( c_1, \ldots, c_N \); \( \text{diag}(c_1 Q_1, \ldots, c_N Q_N) \) denotes the blockwise diagonal matrix with diagonal blocks \( c_1 Q_1, \ldots, c_N Q_N \).

**Proof** We use Theorem 4.41. Let \( X_j^t = X_{t e^j} \). Let \( \psi_X(z) = (\log \hat{\rho})(z) \) with \( \rho = \mathcal{L}(X_j^t) \) for \( z \in \mathbb{R}^d \), and \( \psi_X(z) = (\psi_X(z))_{1 \leq j \leq N} \). Let \( \mu_j = \mathcal{L}((X_j^t)_j) \in \mathfrak{P}(\mathbb{R}^{d_j}) \). Then it follows from (5.9) that

\[
e^{i \psi_X(z)} = E e^{i (z, X_j^t)} = E e^{i (z_j, (X_j^t)_j)} = \hat{\mu}_j (z_j),
\]

where \( z = (z_j)_{1 \leq j \leq N} \in \mathbb{R}^d \) with \( z_j \in \mathbb{R}^{d_j} \). Thus

\[
\psi_X(z) = (\log \hat{\mu}_j (z_j))_{1 \leq j \leq N}.
\]

We have

\[
\hat{\mu}_j (z) a = \hat{\mu}_j (a Q_j^T z), \quad a > 0
\]

by the strict \( Q_j \)-stability of \( \mu_j \). Hence

\[
a^C \psi_X(z) = (a^C (\log \hat{\mu}_j (z_j)))_{1 \leq j \leq N} = \psi_X(b^C (a Q_j^T z))_{1 \leq j \leq N}.
\]

(i) Assume \( \{Z_t : t \geq 0\} \) is \( C \)-selfdecomposable. Let \( \Psi_Z \) be the function \( \Psi \) in (4.11) for \( \{Z_t\} \). For \( b > 1 \) and \( w = (w_j)_{1 \leq j \leq N} \in \mathbb{C}^N \) with \( \text{Re} w_j \leq 0 \), Define \( \Psi_{Z,b}(w) \) by

\[
\Psi_Z(w) = \Psi_Z(b^{-C} w) + \Psi_{Z,b}(w).
\]

Similarly to the proof of Proposition 1.13, we can show that \( e^{\Psi_{Z,b}(iu)} \), \( u \in \mathbb{R}^N \), is an infinitely divisible characteristic function. Further, as in Lemma 5.8, there is an \( \mathbb{R}_+^N \)-valued subordinator \( \{Z_t^{(b)}\} \) such that \( E e^{i (u, Z_t^{(b)})} = e^{i \Psi_{Z,b}(iu)} \). In the proof note that \( \gamma_0^0 = (I - b^{-C}) \gamma_0^0 = \text{diag}(1 - b^{-c_1}, \ldots, 1 - b^{-c_N}) \gamma_0^0 \in \mathbb{R}_+^N \). Now we have

\[
E e^{i (z, Y_t)} = e^{i \Psi_Z (\psi_X(z))} = e^{i \Psi_Z (b^{-C} \psi_X(z))} e^{i \Psi_{Z,b}(\psi_X(z))}
\]

and

\[
b^{-C} \psi_X(z) = (\log \hat{\mu}_j (b^{-C} Q_j^T z))_{1 \leq j \leq N} = \psi_X(b^{-D^T} z)
\]

by (5.10), since
\[ b^{-D^T} z = \text{diag}(b^{-c_1 Q_1^T}, \ldots, b^{-c_N Q_N^T}) z = (b^{-c_j Q_j^T} z)_{1 \leq j \leq N}. \]

Hence
\[ E e^{i(z, Y_t)} = \left( E \exp(i \langle b^{-D^T} z, Y_t \rangle) \right) e^{i \Psi_Z(b \langle z \rangle)}. \]

As the second factor in the right-hand side is the characteristic function of a subordinated process, we see that \( \mathcal{L}(Y_t) \) is \( D \)-selfdecomposable.

(ii) By induction similar to (ii) of Theorem 5.9.

(iii) Assume that \( \{Z_t\} \) is strictly \( C \)-stable, that is, \( a \Psi_Z(w) = \Psi_Z(a C w) \). Then, for \( a > 0 \),
\[ E e^{i(z, Y_{at})} = e^{a \Psi_Z(\psi_X(z))} = e^{a \Psi_Z(a C \psi_X(z))} \]
and, as above,
\[ a C \psi_X(z) = \psi_X(a D^T z). \]

Hence
\[ E e^{i(z, Y_{at})} = E \exp(i \langle a D^T z, Y_t \rangle), \]

which shows \( D \)-stability of \( \{Y_t\} \).

\[ \blacksquare \]

**Remark 5.26** Let \( Q \in M_{(0, \infty)}(d) \) and let
\[ S_Q = \{ \xi \in \mathbb{R}^d : |\xi| = 1, \text{ and } |r Q \xi| > 1 \text{ for every } r > 1 \}. \]

Then any \( x \in \mathbb{R}^d \setminus \{0\} \) is uniquely expressed as \( x = r Q \xi \) with \( r > 0 \) and \( \xi \in S_Q \). Notice that \( S_I \) is the unit sphere \( S \) but \( S_Q \subsetneq S \) for some \( Q \). Let \( \mu \in ID \) with generating triplet \((A, \nu, \gamma)\). Then \( \mu \in L_0(Q) \) if and only if \( QA + AQ^\top \) is nonnegative-definite and
\[ v(B) = \int_{S_Q} \lambda(d\xi) \int_0^\infty 1_B(r Q \xi) \frac{k_\xi(r)}{r} dr, \quad B \in B(\mathbb{R}^d), \]
where \( \lambda \) is a finite measure on \( S_Q \) and \( k_\xi(r) \) is nonnegative, decreasing in \( r \in (0, \infty) \), and measurable in \( \xi \in S_Q \). Under the assumption that \( \alpha > 0 \), \( Q \in M_{(\alpha/2, \infty)}(d) \), and \( \mu \) is purely non-Gaussian, we can show that \( \mu \in \mathcal{G}_{\alpha^{-1} Q} \) if and only if
\[ v(B) = \int_{S_Q} \lambda(d\xi) \int_0^\infty 1_B(r Q \xi) r^{-\alpha^{-1}} dr, \quad B \in B(\mathbb{R}^d), \]
where \( \lambda \) is a finite measure on \( \mathbb{S}_Q \); this statement does not exclude the possibility that \( \mathcal{S}_{(\alpha - 1)Q} \) is the set of trivial distributions. It follows that, if \( \{Z_t\} \) is a non-trivial \((\alpha^{-1}Q)\)-stable \( \mathbb{R}^d_+ \)-valued subordinator, then \( Q \) is strongly restricted. For example then, under the additional assumption that \( d = 2 \) and \( Q \) is of the real Jordan normal form, \( Q \) cannot be

\[
\begin{pmatrix}
q_1 & 1 \\
0 & q_1
\end{pmatrix}
or
\begin{pmatrix}
q_1 & -q_2 \\
q_2 & q_1
\end{pmatrix}
\]

with \( q_1 > 0, q_2 > 0 \) and thus \( Q \) must be of the form

\[
\begin{pmatrix}
q_1 & 0 \\
0 & q_2
\end{pmatrix}
\]


Notes

Halgreen [30] (1979) and Ismail and Kelker [36] (1979) proved assertion (i) of Theorem 5.1 in the case where \( \{X_t\} \) is Brownian motion on \( \mathbb{R} \). Assertion (iii) of Theorem 5.1 was essentially known to Bochner [14] (1955). Theorem 5.25 was given in Barndorff-Nielsen et al. [8] (2001), but we have given a simpler proof. Assertion (ii) of Theorem 5.1 is a special case of Theorem 5.25 (ii) with \( N = 1 \) and \( Q = Q_1 = (1/\alpha)I \). Theorem 5.9 was shown by Pedersen and Sato [71] (2003) for subordination of cone-parameter convolution semigroups on \( \mathbb{R}^d \).


Theorems 5.9 and 5.10 were extended to subordination of cone-parameter convolution semigroups, respectively, by Pedersen and Sato [71] (2003) and by Sato [100] (2009).

Theorem 5.10 was proved in Sato [94] (2001a). Earlier Halgreen [30] (1979) and Shanbhag and Sreehari [112] (1979) proved it under the condition that \( L(Z_1) \) is a generalized \( \Gamma \)-convolution. Remark 5.12 is by Takano [119] (1989,1990).

Characterization and many related results on general distributions in \( \mathcal{S}(Q) \) and \( L_m(Q) \) are discussed in Sharpe [113] (1969), Urbanik [128] (1972a), Sato and Yamazato [111] (1985), and Jurek and Mason [44] (1993). For characterization of distributions in \( \mathcal{S}^0(Q) \), see Sato [87] (1987).

For \( Q \in \mathcal{M}_0(d) \) with \( d \geq 2 \), consider Eq. (2.13) with \( c \) replaced by \( Q \). Then we can extend the notion of Ornstein–Uhlenbeck type process generated by \( \rho \in ID(\mathbb{R}^d) \) and \( c > 0 \) to that generated by \( \rho \) and \( Q \) in a natural way; the extended process is also called Ornstein–Uhlenbeck type process frequently, but let us call it...
as $Q$-OU type process. Connections of distributions in $L_m(Q), m = 0, 1, \ldots, \infty,$ with $Q$-OU type processes are parallel to those of $L_m$ with OU type processes in Chaps. 2 and 3 and the proofs are similar; in fact it was done simultaneously in many papers. However, it was a harder problem to find a criterion of recurrence and transience for $Q$-OU type processes; it was solved by Sato et al. [104] (1996) and Watanabe [133] (1998).